The dual Steenrod algebra and the overlapping shuffle product

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Outline

- Introduction: A topological motivation
- Leibniz-Hopf and Steenrod algebras and their duals
- The main combinatorics: Overlapping shuffle product
- Classical theorems in topology revisited

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Introduction

Motivation from topology

As we do not use any topology later, we can ignore this slide.

The i-th Steenrod operation Sq^i is a series of linear maps

$$H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2) \quad (0 \leq n),$$

which are natural with respect to X and satisfying some conditions.

The set of all Steenrod operations form a non-commutative graded $\mathbb{F}_2$-algebra $\mathcal{A}_2$ under the product given by composition.

Or, more homotopy theoretically

$$\mathcal{A}_2 = H\mathbb{F}_2^* H\mathbb{F}_2 = \bigoplus_{0 \leq i} \lim_{n \to \infty} H^{n+i}(K(\mathbb{F}_2, n); \mathbb{F}_2).$$

General goal

Understanding the structure of $\mathcal{A}_2$ is a classical research theme since it is related to the most important problem in homotopy theory; the *stable homotopy groups of the sphere*.

$\mathcal{A}_2$ has a lot of other applications in algebraic topology and invariant theory.

Problem

Study $\mathcal{A}_2$ purely combinatorially.

To this end, we give an alternative definition of $\mathcal{A}_2$.

The two main characters

Mod-2 Leibniz-Hopf algebra

Let $\mathcal{F}_2$ be the free associative algebra over $\mathbb{F}_2$ generated by the indeterminants $S^1, S^2, S^3, \ldots$ with $|S^i| = i$.
A co-commutative co-product is given by

$$\Delta(S^n) := \sum_{i=0}^{n} S^i \otimes S^{n-i} \quad (S^0 := 1).$$

An $\mathbb{F}_2$-basis is given by

$$\{S^I := S^{i_1} S^{i_2} \cdots S^{i_n} \mid I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n, 0 \leq n < \infty\},$$

where we regard $S^I = 1$ when $n = 0$.

Mod-2 Steenrod algebra

$\mathcal{A}_2$ is defined to be the quotient of $\mathcal{F}_2$ by the ideal generated by the *Adem relations*

$$S^i S^j - \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} S^{i+j-k} S^k \quad (i < 2j)$$

Denote the quotient map by $\pi : \mathcal{F}_2 \to \mathcal{A}_2$ and $Sq^i := \pi(S^i)$.
An $\mathbb{F}_2$-basis is given by

$$\{Sq^J := Sq^{j_1} Sq^{j_2} \cdots Sq^{j_n} \mid J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n_{>0}, j_{k-1} \geq 2j_k \forall k\}.$$

These sequences J are said to be *admissible*.
Ex: $Sq^5 Sq^2 Sq^1$ is admissible while $Sq^3 Sq^2$ is not.

Hopf algebra

A (connected) *Hopf algebra* over a field k is a graded bialgebra A with

- a product $\mu : A \otimes A \to A$
- and a co-product $\Delta : A \to A \otimes A$

satisfying

1. $A = \bigoplus_{i=0}^{\infty} A_i$ such that $A_0 = k$ and $\dim(A_i) < \infty$
2. μ and Δ are associative.
3. Δ is an algebra homomorphism (or equivalently, μ is a co-algebra homomorphism).

The simplest example is the tensor algebra $A_i = V^{\otimes i}$ of a vector space V with $\Delta(v) = v \otimes 1 + 1 \otimes v$.

Both $\mathcal{A}_2$ and $\mathcal{F}_2$ are Hopf algebras.

The duals: $\mathcal{F}_2^*$ and $\mathcal{A}_2^*$

Take the graded $\mathbb{F}_2$-dual of $\pi : \mathcal{F}_2 \to \mathcal{A}_2$ to obtain

$$\pi^* : \mathcal{A}_2^* \hookrightarrow \mathcal{F}_2^*$$

Hence, $\mathcal{A}_2^*$ can be thought of as a sub-Hopf algebra of $\mathcal{F}_2^*$.

Denote the bases dual to S^I (resp. Sq^J) by S_I (resp. Sq_J).
Note that $\pi^*(Sq_J) \neq S_J$ in general !!

A good thing with the dual $\mathcal{F}_2^*$ is that the multiplication is commutative and have a nice combinatorial description as *the overlapping shuffle product*.

Overlapping Shuffle Product

The commutative product in $\mathcal{F}_2^*$ is described as the *overlapping shuffle product (OSP)*. We only look at two examples, which illustrate the definition:

$$S_a S_b = S_{a,b} + S_{b,a} + S_{a+b}$$

To compute the product $S_{a_1,a_2} S_b$, we just look at the indices:

$$(a_1, a_2)(b) = (a_1, a_2, b) + (a_1, b, a_2) + (b, a_1, a_2) + (a_1, a_2 + b) + (a_1 + b, a_2)$$

Remark: $\mathcal{F}_2^*$ is a polynomial algebra generated by elementary concatenation powers of elementary Lyndon words.

Classical theorems revisited

$\mathcal{A}_2^*$ is a polynomial ring

We want to identify the image of the inclusion $\pi^* : \mathcal{A}_2^* \hookrightarrow \mathcal{F}_2^*$.

Set $\xi_n := S_{2^{n-1},2^{n-2},\ldots,2^0}$. The degree is $|\xi_n| = 2^n - 1$.

Theorem

The image of π^* is a polynomial ring generated by ξ_n's.

Thus, we rediscover a famous theorem of Milnor:

Corollary

$$\mathcal{A}_2^* \simeq \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

The key point is that there is an ordering on the bases S_I compatible with OSP:

Change of basis

Since $\mathcal{A}_2^* \simeq \mathbb{F}_2[\xi_1, \xi_2, \ldots]$, it is given another basis consisting of monomials in ξ_n. This basis is called the (dual) *Milnor basis*.
A question is, how to compare it with the basis consisting of Sq_J's.

We have a linear left inverse $r\colon \mathcal{F}_2^* \to \mathcal{A}_2^*$ of π^* such that $r \circ \pi^* = 1$:

$$r(S_I) = \begin{cases} Sq_I & (I : \text{admissible}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Combined with $\xi_n = S_{2^{n-1}, 2^{n-2}, \ldots, 2^0}$, we can compute the basis change. For example,

$$r(\xi_1^2 \xi_2) = r(S_2 S_{2,1}) = r(S_{4,1} + S_{2,1,2} + S_{2,3}) = Sq_{4,1}$$
$$r(\xi_1^5) = r(S_1 S_4) = r(S_5 + S_{4,1} + S_{1,4}) = Sq_5 + Sq_{4,1}$$

Conjugation

It is known that any connected (co)commutative Hopf algebra has a *conjugation* χ satisfying

1. $\chi(1) = 1$
2. $\sum x'\chi(x'') = 0$, where $\Delta(x) = \sum x' \otimes x''$
3. $\chi(xy) = \chi(y)\chi(x)$
4. $\chi^2(x) = x$

Conjugation is uniquely characterised by the first two conditions.

Problem

Give formulae for the conjugations in $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$.

Conjugation in $\mathcal{F}_2^*$

We give two formulae for the conjugation in $\mathcal{F}_2^*$:

Theorem

$$\chi(S_I) = \sum_{I' \in C(I^{-1})} S_{I'},$$

where $C(I^{-1})$ is the coarsening of the reverse word of I.

$$\chi(S_I) = \sum_{\beta \in \mathcal{P}(I)} \prod_{k=1}^{l(\beta)} S_{\beta(k)},$$

where $\mathcal{P}(I)$ is the set of ordered partition of I.

Ex: $\chi(S_{1,2,3}) = S_{3,2,1} + S_{5,1} + S_{3,3} + S_6$ (by the first formula)
$= S_{1,2,3} + S_1 S_{2,3} + S_{1,2} S_3 + S_1 S_2 S_3$ (by the second formula)

Conjugation in $\mathcal{A}_2^*$

Our second formula is a generalisation of Milnor's formula for the conjugation in $\mathcal{A}_2^*$:

Corollary

$$\chi(\xi_n) = \sum_{\alpha} \prod_{k=1}^{l(\alpha)} \xi_{\alpha(k)}^{2^{\sigma(k)}},$$

Ours is a lift of Milnor's through $\mathcal{A}_2^* \subset \mathcal{F}_2^*$.

Topological remark: In view of $\mathcal{A}_2^* = (H\mathbb{F}_2)_* H\mathbb{F}_2 = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$, the conjugation is induced by the switching of two factors. Hence, it is related to the commutativity of the spectrum $H\mathbb{F}_2$.

Did we forget $\mathcal{A}_2$ and $\mathcal{F}_2$?

Duality

We can think of a sequence I as a string $(1 * 1 * \cdots * 1)$, where $*$ is either "$+$" or the comma ",".

Definition

The dual $\bar{I}$ of I is the string obtained by switching $+$ and the commas.

Ex: For $I = (1, 3, 2) = (1, 1 + 1 + 1, 1 + 1)$, its dual is

$$\bar{I} = (1 + 1, 1, 1 + 1, 1) = (2, 1, 2, 1).$$

It is easily seen that $\bar{\bar{I}} = I$ and $I \prec I' \Leftrightarrow \bar{I} \succ \bar{I}'$.

Extend the duality to one between $\mathcal{F}_2$ and $\mathcal{F}_2^*$ by

$$D(S^I) = S_{\bar{I}}, \quad D^{-1}(S_I) = S^{\bar{I}}.$$

Duality

This duality map $D : \mathcal{F}_2 \leftrightarrow \mathcal{F}_2^*$ commutes with the conjugation

Theorem

$$D \circ \chi = \chi \circ D$$

In particular, $f \in \mathcal{F}_2$ is a conjugation invariant iff so is $\bar{f} \in \mathcal{F}_2^*$.

Note that the submodule consisting of conjugation invariants $\chi(x) = x$ is actually a subalgebra. This subalgebra for $\mathcal{F}_2^*$ and $\mathcal{A}_2^*$ has been studied by Crossley, Whitehouse, and Turgay.

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Maple code

Our method works with any $\mathbb{F}_p$ with a prime p if we consider *Bockstein free part* of the mod-p Steenrod algebra.

It is made into a Maple code, which can compute

- Basis change between ξ^I and $Sq^I \quad (\mathcal{P}^I)$
- Adem relations
- Conjugations in $\mathcal{F}_p, \mathcal{A}_p$ and their duals
- Conjugation invariants

The code is available at http://skaji.org/code/

Future work

- Understand Adem relations by determining π^*
- Basis change with respect to other basis of $\mathcal{A}_2^*$
- Determine the χ-invariant subring of $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$
- Extend our results to mod-p Steenrod algebra (not only for Bockstein free part)

Thank you very much

Teşekkürler !