Products in equivariant homology

Shizuo KAJI
Yamaguchi Univ. Japan / U. of Southampton UK

joint with Haggai Tene

UAB Topology Seminar
Barcelona, 12 June, 2015

Outline

Let h be a ring spectrum. The goal of this talk is to generalise the cross product in the homology

$$\times : h_s(X) \otimes h_t(Y) \to h_{s+t}(X \times Y).$$

Outline of talk:

1. Two external products for a fibre square
2. An equivariant external product for a fibre square
3. Vanishing and a secondary product
4. Computational examples

Preprint at arXiv:1506.00441

External product for fibre square

Given a homotopy pullback

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

with certain conditions on B.
We will define homomorphisms in the form

$$h_s(X) \otimes h_t(Y) \to h_{s+t+*}(X \times_B Y)$$

with degree shifts, which reduce to the ordinary cross product when $B = pt$.

External product for fibre square

The key idea is that a homotopy pullback

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

is equivalent to the pullback of fibrations

$$\begin{array}{ccccc} \Omega B & \longrightarrow & X \times_B Y & \xrightarrow{\hat{\Delta}} & X \times Y \\ \| & & \downarrow & & \downarrow f \times g \\ \Omega B & \longrightarrow & B & \xrightarrow{\Delta} & B \times B \end{array}$$

and we use *wrong-way maps* $\hat{\Delta}^* : h_*(X \times X) \to h_{*+\mathrm{shift}}(X \times_B Y)$.

External product of first type

External product for fibre square

We define two kinds of external products in different settings.
First, let $B = BG$ be the classifying space of a compact Lie group G.
We require a technical condition: the universal adjoint bundle $\mathfrak{g} \hookrightarrow ad(EG) \to BG$ is oriented.
Consider the pullback diagram

$$\begin{array}{ccccc}
\Omega BG & \longrightarrow & X \times_{BG} Y & \xrightarrow{\hat{\Delta}} & X \times Y \\
\| & & \downarrow & & \downarrow \\
\Omega BG & \longrightarrow & BG & \xrightarrow{\Delta} & BG \times BG \\
\| & & \| & & \| \\
(G \times G)/\Delta G & \longrightarrow & E(G \times G)/\Delta G & \longrightarrow & E(G \times G)/(G \times G)
\end{array}$$

Grothendieck bundle transfer

We need an wrong-way map for $\hat{\Delta} : X \times_{BG} Y \to X \times Y$.

Let $F \to E \to B$ be a fiber bundle with a compact Lie structure group, where the fibre F is a compact manifold. By fibre-wise Pontrjagin-Thom construction, we obtain a map between Thom spectra $B^0 \to E^{-t}$, where t is the bundle of tangents. Applying this to our setting, we have

$$\hat{\Delta}^{\natural} : h_{s+t}(X \times Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y).$$

Composing with the cross product, we define

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

Application: String product in BG

Consider the following pullback diagram

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, BG) & \longrightarrow & LBG \\ \downarrow & & \downarrow{\scriptstyle ev} \\ LBG & \xrightarrow{ev} & BG \end{array}$$

where $LBG \simeq \mathrm{Map}(S^1, BG) \simeq BLG$.
Composing μ_{BG} with the concatenation of loops
$\mathrm{Map}(S^1 \vee S^1, BG) \to LBG$, we obtain

$$h_s(LBG) \otimes h_t(LBG) \to h_{s+t+\dim(G)}(LBG)$$

which is equivalent to the Chataur-Menichi product when $h_* = H_*$.

External product of second type

External product for fibre square

Second, let $B = M$ be an oriented closed manifold. Consider the pullback diagram:

$$\begin{array}{ccc} X \times_M Y & \xrightarrow{\hat{\Delta}} & X \times Y \\ \downarrow & & \downarrow f \times g \\ M & \xrightarrow{\Delta} & M \times M. \end{array}$$

The diagonal $\Delta : M \to M \times M$ is a finite codimensional embedding with the normal bundle isomorphic to TM. The Gysin map $\Delta^! : h_*(M \times M) \to h_{*-\mathrm{dim}(M)}(M)$ is the Poincare dual of the cup product.
We can pull it back to define a Gysin map for $\hat{\Delta}$:

$$\hat{\Delta}^! : h_*(X \times Y) \to h_{*-\mathrm{dim}(M)}(X \times_M Y).$$

External product for fibre square

Composing

$$\hat{\Delta}^! : h_{s+t}(X \times Y) \to h_{s+t-\dim(M)}(X \times_M Y).$$

with the cross product, we define

$$\mu_M : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_M Y).$$

Note that when $X = Y = M$ and $h_* = H_*$, it restricts to the intersection product

$$H_s(M) \otimes H_t(M) \to H_{s+t-\dim(M)}(M).$$

Application: String product in M

Consider the following pullback

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, BG) & \longrightarrow & LM \\ \downarrow & & \downarrow{\scriptstyle ev} \\ LM & \xrightarrow{ev} & M \end{array}$$

Composing μ_M with the concatenation of loops, we obtain

$$h_s(LM) \otimes h_t(LM) \to h_{s+t-\dim(M)}(LM),$$

which is equivalent to the Chas-Sullivan product when $h_* = H_*$.

Two external products

When $B = BG$ is the classifying space of a compact Lie group

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

When $B = M$ is a closed oriented manifold

$$\mu_M : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_M Y)$$

Can we unify the two constructions?

Equivariant external product

Equivariant external product

We combine the previous two constructions and define an external product in an equivariant setting. Let G acts on M orientation preservingly and M_G be the Borel construction. For a homotopy pullback

$$\begin{array}{ccc} X \times_{M_G} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M_G \end{array}$$

we will define

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

Note that any map $f : X \to M_G$ is converted to an equivariant map by considering the pullback:

$$\begin{array}{ccccc} \hat{X} & \xrightarrow{\hat{f}} & M & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & M_G & \longrightarrow & BG, \end{array}$$

which identifies $f = \hat{f}_G : \hat{X}_G \to M_G$.

Hence, the initial diagram is equivalent to the Borel construction of

$$\begin{array}{ccc} \hat{P} & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ \hat{Y} & \longrightarrow & M \end{array}$$

Definition: Equivariant product

The problem to define

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

is that we cannot define a wrong-way map for the diagonal map

$$M_G \to M_G \times M_G$$

since its fibre is not finite dimensional and it is not a finite codimensional embedding.

We will decompose the diagonal map into two steps and define wrong-way maps step-by-step.

Definition: Equivariant product

The diagonal map $M_G \to M_G \times M_G$ decomposes into two steps:

$$M_G \xrightarrow{\Delta_G} (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G,$$

where Δ_G is the equivariant diagonal (which is codimension $\dim(M)$) and

$$(G \times G)/\Delta G \hookrightarrow (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G$$

is the homogeneous fibration. That is, the pullback of

$$(G \times G)/\Delta G \hookrightarrow E(G \times G)/\Delta G \to E(G \times G)/(G \times G)$$

Definition: Equivariant product

Define Q as in the ladder of pullbacks:

$$\begin{array}{ccccc} X \times_{M_G} Y & \xrightarrow{\hat{\Delta}_G} & Q & \xrightarrow{\hat{q}} & X \times Y \\ \downarrow & & \downarrow & & \downarrow \\ M_G & \xrightarrow{\Delta_G} & (M \times M)_G & \xrightarrow{q} & M_G \times M_G. \end{array}$$

Then define the product μ_{M_G} as the composition:

$$h_s(X) \otimes h_t(Y) \to h_{s+t}(X \times Y)$$
$$\xrightarrow{\hat{q}^\natural} h_{s+t+\dim(G)}(Q) \xrightarrow{\hat{\Delta}_G^!} h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

Equivariant product unifies the two

The equivariant external product

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y)$$

restricts to

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

when $M = pt$ and

$$\mu_M : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_M Y).$$

when $G = *$.

Properties of μ_{M_G}

Proposition

- Natural with respect to a homomorphism between homology theories
- Compatible with the group restriction: Let $H \subset G$ be a closed subgroup and $i : M_H \to M_G$ be the induced map. Then, $\mu_{M_H} \circ (i^\natural \otimes i^\natural) = i^\natural \circ \mu_{M_G}$.

$$\begin{array}{ccccccc} X' \times_{M_H} Y' & \longrightarrow & X' & \xrightarrow{i} & X \times_{M_G} Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & M_H & & Y & \longrightarrow & M_G \end{array}$$

- Compatible with the induction $(M \times G/H)_G \simeq M_H$.

Vanishing of μ_{M_G}

From now on, we specialises the case when $h_* = H_*$.
An easy but interesting property of μ_{M_G} is that it vanishes in higher degrees:

Theorem

Let F_X (resp. F_Y) be the homological dimension of the homotopy fibre of the composition $X \to M_G \to BG$. Then,

$$\mu_{M_G} : H_s(X; R) \otimes H_t(Y; R) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y; R)$$

vanishes if $s > F_X - \dim(G)$ or $t > F_Y - \dim(G)$.

When applied to special cases, it has non-trivial consequences.

Application of the external product

Chataur-Menichi defined *string operations* for LBG:
For a surface $F_{g,p+q}$ of genus g with *p-incoming* and *q-outgoing* boundary circles, they defined a homomorphism

$$\mu(F_{g,p+q}) : H_*(LBG)^{\otimes p} \to H_{\dim(G)(2g+p+q-2)}(LBG)^{\otimes q}$$

which is compatible with the gluing of the surfaces.
(when $g = 0, p = 2, q = 1$, it gives the product we saw earlier)
A consequence of our vanishing theorem is:

Corollary

$\mu(F_{g,p+q})$ is trivial unless $g = 0$ and $p = 1$, or $* = 0$.

Secondary product

Secondary product

Vanishing of

$$\mu_{M_G} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

for $s > F_X - \dim(G)$ or $t > F_Y - \dim(G)$ suggests that we may define a "secondary" product.
In fact, we can define

$$\psi : H_s(X) \otimes H_t(Y) \to H^G_{s+t+\dim(G)-\dim(M)+1}(X \times_{M_G} Y).$$

for $s > F_X - \dim(G)$ and $t > F_Y - \dim(G)$.

Given

$$\begin{array}{ccc} X \times_{M_{\boldsymbol{G}}} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M_G \end{array}$$

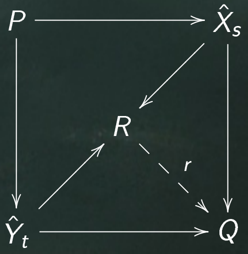
consider the following diagram with all the squares pullback:

$$\begin{array}{ccccc} Z & \longrightarrow & \hat{X}_s & \longrightarrow & X_s \\ \downarrow & & \downarrow & & \downarrow \\ \hat{Y}_t & \longrightarrow & Q & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y_t & \longrightarrow & Y & \longrightarrow & BG, \end{array}$$

where X_s and Y_t are s- and t-skeleta, and Q is same as in the definition of $\mu_{M_{\boldsymbol{G}}}$.

Secondary product

Now we take the homotopy pushforward of the upper-left corner



Since $\hat{X}_s$ and $\hat{Y}_t$ have low homological degrees by assumption, we obtain a well-defined map by the composition

$$H_*(P) \xrightarrow{\text{sus}} H_{*+1}(R) \xrightarrow{r_*} H_{*+1}(Q)$$

Secondary product

Our secondary product is defined to be the composition

$$\psi : H_s(X) \otimes H_t(Y) \xrightarrow{\text{lift}} H_s(X_s) \otimes H_t(Y_t) \xrightarrow{\mu_{BG}} H_{s+t+\dim(G)}(P)$$

$$\to H_{s+t+\dim(G)+1}(Q) \xrightarrow{\hat{\Delta}^!_G} H_{s+t+\dim(G)-\dim(M)+1}(X \times_{M_G} Y).$$

This can be shown to be well-defined. (it does not depend on the choices of skeleta and lift of classes)

Proposition

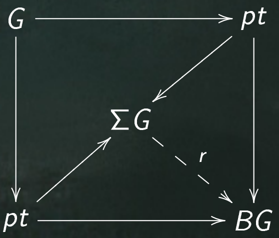
ψ is compatible with restriction wrt a closed subgroup $H \subset G$.

Simplest example

Let $X = Y = M = pt$. Then the defining pull-push diagram for

$$\psi : H_0(pt) \otimes H_0(pt) \to H_{\dim(G)+1}(BG)$$

is identified with the first stage of the Ganea construction:



Application: String Product in BG

Again from the diagram

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, BG) & \longrightarrow & LBG \\ \big\downarrow & & \big\downarrow ev \\ LBG & \xrightarrow{ev} & BG \end{array}$$

we obtain

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)+1}(LBG),$$

which is a secondary product of Chataur-Menichi's string product.
This product does not usually vanish!

Application: Intersection product

When $X = Y = M_G$, that is, for the identity square

$$\begin{array}{ccc} M_G & = & M_G \\ \| & & \| \\ M_G & = & M_G, \end{array}$$

ψ specialises to a product in $H_*^G(M)$:

$$\psi : H_s^G(M) \otimes H_t^G(M) \to H_{s+t+\dim(G)-\dim(M)+1}^G(M),$$

which can be thought of as a secondary equivariant intersection product.

Application: Tate cohomology

When $M = pt$, we have

$$H_s(BG) \otimes H_t(BG) \to H_{s+t+\dim(G)+1}(BG)$$

Theorem

It coincides with the product in Tate cohomology when G is finite.

That is, ψ generalises Tate cohomology ring in two ways:

- G can now be not only a finite group but a compact Lie group
- the group homology (the equivariant homology of a point) is replaced by the equivariant homology of a manifold

Computational Examples

First computation

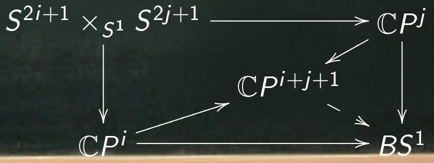
When $G = S^1, M = pt$

$$H_*^G(pt; \mathbb{Z}) = H_*(BS^1; \mathbb{Z}) \simeq \mathbb{Z}\langle a_{2k} \rangle \quad (k \geq 0),$$

where a_{2k} is represented by $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^\infty = BS^1$.

The product $H_s(BS^1) \otimes H_t(BS^1) \to H_{s+t+2}(BS^1)$ is given by

$$a_{2i} * a_{2j} = a_{2(i+j+1)}.$$



Computation for classical groups

Proposition

The product vanishes for all compact connected classical Lie groups of rank greater than 1.

For $H_*(BSp(1); \mathbb{Z}) \simeq \mathbb{Z}\langle a_{4k} \rangle \quad (k \geq 0)$,

$$a_{4i} * a_{4j} = a_{4(i+j+1)}.$$

For $H_*(BSO(3); \mathbb{Z}) \simeq \mathbb{Z}\langle b_{4k} \rangle \oplus 2\text{-torsion} \quad (k \geq 0)$,

$$b_{4i} * b_{4j} = 2b_{4(i+j+1)}$$

and all the other products vanish.

Computation for $\mathbb{C}P^1$

Let $S^1 \curvearrowright \mathbb{C}P^1$ by the standard action. Then, $H_*^{S^1}(\mathbb{C}P^1) = \mathbb{Z}\langle \alpha_{2k}, \beta_{2k+2} \rangle$, where

$$\alpha_{2k} : S^{2n+1} \times_{S^1} pt \to ES^1 \times_{S^1} \mathbb{C}P^1$$
$$\beta_{2k+2} : S^{2n+1} \times_{S^1} \mathbb{C}P^1 \to ES^1 \times_{S^1} \mathbb{C}P^1$$

$\psi : H_s^{S^1}(\mathbb{C}P^1) \otimes H_t^{S^1}(\mathbb{C}P^1) \to H_{s+t}^{S^1}(\mathbb{C}P^1)$ is computed as

Proposition

$\alpha_{2i} * \alpha_{2j} = 0, \alpha_{2i} * \beta_{2j+2} = \alpha_{2(i+j+1)}, \beta_{2i+2} * \beta_{2j+2} = \beta_{2(i+j+1)+2}$

Future work

- Find other applications (e.g., obstruction for group action)
- Develop computational method (e.g., Eilenberg-Moore SS)
  - secondary product for $H_*(BG)$
  - secondary product for $H_*^T(M)$ for toric and flag manifolds
- Relation with other structures (co-product, Steenrod co-operations)
- Compare the secondary product to Greenlees-May's Tate cohomology
- Extend to more general groups (e.g., p-compact groups)

Moltes grácies !