# The dual Steenrod algebra and the overlapping shuffle product

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January 18, 2016

# Outline

- Introduction: A topological motivation
- Leibniz-Hopf and Steenrod algebras and their duals
- The main combinatorics: Overlapping shuffle product
- Classical theorems in topology revisited

Our preprint is available at arXiv:1505.05465

# Motivation from topology

The $i$-th Steenrod operation $Sq^i$ is a series of linear maps

$$H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2) \quad (0 \le n),$$

which are natural with respect to $X$ and satisfying some conditions.

The set of all Steenrod operations form a non-commutative graded $\mathbb{F}_2$-algebra $\mathcal{A}_2$ under the product given by composition.

Or, more homotopy theoretically

$$\mathcal{A}_2 = H\mathbb{F}_2^* H\mathbb{F}_2 = \bigoplus_{0 \le i} \lim_{n\to\infty} H^{n+i}(K(\mathbb{F}_2, n);\mathbb{F}_2).$$

# General goal

An $\mathbb{F}_2$-basis of $\mathcal{A}_2$ is given by the *admissible elements*

$$\{Sq^I := Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n} \mid I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n, 0 \leq n < \infty\}$$
$$(Sq^{\emptyset} = 1)$$

$\mathcal{A}_2$ admits a co-commutative co-product and is a *Hopf algebra*:

$$\Delta Sq^k = \sum_{i+j=k} Sq^i \otimes Sq^j \qquad \text{(Cartan's formula)}$$

Its dual Hopf algebra $\mathcal{A}_2^* := \mathrm{Hom}(\mathcal{A}_2, \mathbb{F}_2) = (H\mathbb{F}_2)_* H\mathbb{F}_2$ is commutative. The $\mathbb{F}_2$-basis dual to $Sq^I$ is denoted by $Sq_I$.

## Problem

Study $\mathcal{A}_2$ and $\mathcal{A}_2^*$ purely combinatorially.

# Milnor's Theorems

Set $\xi_n := Sq_{2^{n-1},2^{n-2},\ldots,2^0}$. The degree is $|\xi_n| = 2^n - 1$.

## Theorem

The dual Steenrod algebra is a polynomial ring:

$$\mathcal{A}_2^* \simeq \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

The anti-pode (conjugation) is given by:

$$\chi(\xi_n) = \sum_{\alpha} \prod_{k=1}^{l(\alpha)} \xi_{\alpha(k)}^{2^{\sigma(k)}}$$

# Mod-2 Leibniz-Hopf algebra

Let $\mathcal{F}_2$ be the free associative algebra over $\mathbb{F}_2$ generated by the indeterminants $S^1, S^2, S^3, \ldots$ with $|S^i| = i$.
A co-commutative co-product is given by

$$\Delta(S^n) := \sum_{i=0}^{n} S^i \otimes S^{n-i} \quad (S^0 := 1).$$

An $\mathbb{F}_2$-basis is given by

$$\{S^I := S^{i_1} S^{i_2} \cdots S^{i_n} \mid I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n, 0 \leq n < \infty\},$$

where we regard $S^I = 1$ when $n = 0$.

Note: $\dfrac{\partial^n}{(\partial x)^n}(f \cdot g) = \sum_{i=0}^{n} \binom{n}{i} \dfrac{\partial^i}{(\partial x)^i}(f) \dfrac{\partial^{n-i}}{(\partial x)^{n-i}}(g)$.

# Mod-2 Steenrod algebra

$\mathcal{A}_2$ is defined to be the quotient of $\mathcal{F}_2$ by the ideal generated by the *Adem relations*

$$S^i S^j - \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} S^{i+j-k} S^k \quad (i < 2j)$$

Denote the quotient map by $\pi : \mathcal{F}_2 \to \mathcal{A}_2$ and $Sq^i := \pi(S^i)$.
An $\mathbb{F}_2$-basis is given by

$$\{Sq^J := Sq^{j_1} Sq^{j_2} \cdots Sq^{j_n} \mid J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n_{>0}, j_{k-1} \geq 2j_k \forall k\}.$$

These sequences $J$ are said to be *admissible*.
Ex: $Sq^5 Sq^2 Sq^1$ is admissible while $Sq^3 Sq^2$ is not.

# Hopf algebra

A (connected) *Hopf algebra* over a field $k$ is a graded bialgebra $A$ with

- a product $\mu : A \otimes A \to A$
- and a co-product $\Delta : A \to A \otimes A$

satisfying

1. $A = \bigoplus_{i=0}^{\infty} A_i$ such that $A_0 = k$ and $\dim(A_i) < \infty$
2. $\mu$ and $\Delta$ are associative.
3. $\Delta$ is an algebra homomorphism (or equivalently, $\mu$ is a co-algebra homomorphism).

The simplest example is the tensor algebra $A_i = V^{\otimes i}$ of a vector space $V$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$.

Both $\mathcal{A}_2$ and $\mathcal{F}_2$ are Hopf algebras.

# The duals: $\mathcal{F}_2^*$ and $\mathcal{A}_2^*$

Take the graded $\mathbb{F}_2$-dual of $\pi : \mathcal{F}_2 \to \mathcal{A}_2$ to obtain

$$\pi^* : \mathcal{A}_2^* \hookrightarrow \mathcal{F}_2^*$$

Hence, $\mathcal{A}_2^*$ can be thought of as a sub-Hopf algebra of $\mathcal{F}_2^*$.

Denote the bases dual to $S^I$ (resp. $Sq^J$) by $S_I$ (resp. $Sq_J$).
Note that $\pi^*(Sq_J) \neq S_J$ in general !!

$$\begin{aligned} Sq^I &= \sum_{J:\text{admissible}} C^I_J Sq^J \\ \Leftrightarrow \pi^*(Sq_J) &= \sum_{I:\text{sequence}} C^I_J S_I \end{aligned}$$

A good thing with the dual $\mathcal{F}_2^*$ is that the multiplication is commutative

and have a nice combinatorial description as *the overlapping shuffle product*.

# Overlapping Shuffle Product

Note: $S^I S^J = S^{I,J}$ but $S_I S_J \neq S_{I,J}$!

The commutative product in $\mathcal{F}_2^*$ is described as the *overlapping shuffle product (OSP)*. We only look at two examples, which illustrate the definition:

$$S_a S_b = S_{a,b} + S_{b,a} + S_{a+b}$$

To compute the product $S_{a_1,a_2} S_b$, we just look at the indices:

$$\begin{aligned}(a_1, a_2)(b) = (a_1, a_2, b) + (a_1, b, a_2) + (b, a_1, a_2)\\ + (a_1, a_2 + b) + (a_1 + b, a_2)\end{aligned}$$

Remark: $\mathcal{F}_2^*$ is a polynomial algebra generated by elementary concatenation powers of elementary Lyndon words.

# Classical theorems revisited

# $\mathcal{A}_2^*$ is a polynomial ring

We can show $\pi^*(Sq_{2^n}) = S_{2^n}, \quad \pi^*(Sq_{2^{n-1},2^{n-2},\ldots,2^0}) = S_{2^{n-1},2^{n-2},\ldots,2^0}$.
We want to identify the image of the inclusion $\pi^* : \mathcal{A}_2^* \hookrightarrow \mathcal{F}_2^*$.

Set $\xi_n := S_{2^{n-1},2^{n-2},\ldots,2^0}$. The degree is $|\xi_n| = 2^n - 1$.

**Theorem**

The image of $\pi^*$ is a polynomial ring generated by $\xi_n$'s.

Thus, we rediscover a famous theorem of Milnor:

**Corollary**

$$\mathcal{A}_2^* \simeq \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

# Sketch of proof

The key point is that there is an ordering on the bases $S_I$ compatible with OSP:

$$(a_1,a_2,\ldots,a_n)>(b_1,b_2,\ldots,b_m) \text{ iff } (n>m) \text{ or } (\exists k, a_k>b_k \text{ and } a_i=b_i \forall i>k),$$

The lowest term in $S_I \cdot S_{I'}$ for $I=(i_1,i_2,\ldots)$ is $S_{(i_1+i'_1,i_2+i'_2,\ldots)}$. Then, by the *excess bijection* between admissible sequences and sequences of non-negative integers.

$$\gamma(j_1,j_2,\ldots,j_n) := (j_1-2j_2, j_2-2j_3,\ldots,j_{n-1}-2j_n,j_n)$$

we have the upper triangularity

$$\xi^{\gamma(J)} = S_J + (\text{terms higher than } S_J).$$

# Change of basis

Since $\mathcal{A}_2^* \simeq \mathbb{F}_2[\xi_1, \xi_2, \dots]$, it is given another basis consisting of monomials in $\xi_n$. This basis is called the (dual) *Milnor basis*.
A question is, how to compare it with the basis consisting of $Sq_J$'s.

We have a linear left inverse $r\colon \mathcal{F}_2^* \to \mathcal{A}_2^*$ of $\pi^*$ such that $r \circ \pi^* = 1$:

$$r(S_I) = \begin{cases} Sq_I & (I : \text{admissible}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Combined with $\xi_n = S_{2^{n-1}, 2^{n-2}, \dots, 2^0}$, we can compute the basis change. For example,

$$r(\xi_1^2 \xi_2) = r(S_2 S_{2,1}) = r(S_{4,1} + S_{2,1,2} + S_{2,3}) = Sq_{4,1}$$
$$r(\xi_1^5) = r(S_1 S_4) = r(S_5 + S_{4,1} + S_{1,4}) = Sq_5 + Sq_{4,1}$$

# Conjugation

It is known that any connected (co)commutative Hopf algebra has a *conjugation* $\chi$ satisfying

1. $\chi(1) = 1$
2. $\sum x'\chi(x'') = 0$, where $\Delta(x) = \sum x' \otimes x''$
3. $\chi(xy) = \chi(y)\chi(x)$
4. $\chi^2(x) = x$

Conjugation is uniquely characterised by the first two conditions.

**Problem**

Give formulae for the conjugations in $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$.

## Conjugation in $\mathcal{F}_2^*$

A coarsening set $C(I)$ of a sequence $I = (i_1, \ldots, i_l)$ is defined recursively

$$C(I) := \{(i_1, I'), (i_1 + i'_1, I'_2) \mid I' \in C((i_2, \ldots, i_l))\} \text{ and } C((i)) = \{i\},$$

where $I'_2$ is the tail partial sequence $(i'_2, \ldots, i'_{l'})$ of $I' = (i'_1, i'_2, \ldots, i'_{l'})$.

Ex: $C(a, b, c) = \{(a, b, c), (a + b, c), (a, b + c), (a + b + c)\}$.

By checking the characterisation property, we see

$$\chi(S_I) = \sum_{I' \in C(I^{-1})} S_{I'},$$

where $I^{-1} = (i_l, \ldots, i_1)$ is the reverse sequence of $I = (i_1, \ldots, i_l)$.

# Conjugation in $\mathcal{F}_2^*$

Again by the characterisation, we can show an alternative formula:

**Theorem**

$$\chi(S_I) = \sum_{\beta \in \mathcal{P}(I)} \prod_{k=1}^{l(\beta)} S_{\beta(k)},$$

where $\mathcal{P}(I)$ is the set of ordered partition of $I$.

Ex:

$$\begin{aligned} \chi(S_{1,2,3}) &= S_{3,2,1} + S_{5,1} + S_{3,3} + S_6 \text{ (by the first formula)} \\ &= S_{1,2,3} + S_1 S_{2,3} + S_{1,2} S_3 + S_1 S_2 S_3 \text{ (by the second formula)} \end{aligned}$$

# Conjugation in $\mathcal{A}_2^*$

Our new formula is a generalisation of Milnor's formula for the conjugation in $\mathcal{A}_2^*$:

**Corollary**

$$\chi(\xi_n) = \sum_{\alpha} \prod_{k=1}^{l(\alpha)} \xi_{\alpha(k)}^{2^{\sigma(k)}},$$

where $\alpha$ runs through ordered partitions of $n$ and $\sigma(k) = \sum_{j=1}^{k-1} \alpha(j)$.

Ours is a lift of Milnor's through $\mathcal{A}_2^* \subset \mathcal{F}_2^*$.

Topological remark: In view of $\mathcal{A}_2^* = (H\mathbb{F}_2)_* H\mathbb{F}_2 = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$, the conjugation is induced by the switching of two factors. Hence, it is related to the commutativity of the spectrum $H\mathbb{F}_2$.

# Duality

We can think of a sequence $I$ as a string $(1 * 1 * \cdots * 1)$, where $*$ is either "$+$" or the comma ",".

**Definition**

The dual $\bar{I}$ of $I$ is the string obtained by switching $+$ and the commas.

Ex: For $I = (1,3,2) = (1, 1+1+1, 1+1)$, its dual is

$$\bar{I} = (1+1, 1, 1+1, 1) = (2,1,2,1).$$

It is easily seen that $\bar{\bar{I}} = I$ and $I \prec I' \Leftrightarrow \bar{I} \succ \bar{I}'$.

Extend the duality to one between $\mathcal{F}_2$ and $\mathcal{F}_2^*$ by

$$D(S^I) = S_{\bar{I}}, \quad D^{-1}(S_I) = S^{\bar{I}}.$$

# Duality

This duality map $D : \mathcal{F}_2 \leftrightarrow \mathcal{F}_2^*$ commutes with the conjugation

Theorem

$$D \circ \chi = \chi \circ D$$

In particular, $f \in \mathcal{F}_2$ is a conjugation invariant iff so is $\bar{f} \in \mathcal{F}_2^*$.

Note that the submodule consisting of conjugation invariants $\chi(x) = x$ is actually a subalgebra. This subalgebra for $\mathcal{F}_2^*$ and $\mathcal{A}_2^*$ has been studied by Crossley, Whitehouse, and Turgay.

# Maple code

Our method works with any $\mathbb{F}_p$ with a prime $p$ if we consider *Bockstein free part* of the mod-$p$ Steenrod algebra.

It is made into a Maple code, which can compute

- Basis change between $\xi^I$ and $Sq^I \quad (\mathcal{P}^I)$
- Adem relations
- Conjugations in $\mathcal{F}_p, \mathcal{A}_p$ and their duals
- Conjugation invariants

The code is available at http://skaji.org/code/

# Future work

- Understand Adem relations by determining $\pi^*$
- Basis change with respect to other basis of $\mathcal{A}_2^*$
- Determine the $\chi$-invariant subring of $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$
- Extend our results to mod-$p$ Steenrod algebra (not only for Bockstein free part)