Mod p decompositions of the loop spaces of flag manifolds

Shizuo KAJI
Yamaguchi Univ. Japan / U. of Southampton UK

joint with A. Ohsita and S. Theriault

International Mathematics Conference
in honour of the 70th Birthday of
Professor S. A. Ilori
Ibadan, 12 Jan. 2015

Outline

- Mod p homotopy decomposition
- Homogeneous spaces and Flag manifolds
- Decomposition of the loop spaces of homogeneous spaces
- Homotopy groups of flag manifolds

General goal

Problem

Compute the homotopy groups $\pi_*(X)$ of flag manifolds X.

Basic properties of $\pi_*(X)$

1. If a topological space X is homotopy equivalent to a product

$$X \simeq X_1 \times X_2 \times \cdots \times X_n,$$

   we can calculate the homotopy groups by

$$\pi_*(X) \simeq \pi_*(X_1) \oplus \cdots \oplus \pi_*(X_n).$$

2. If we have a fibration $X \to E \to B$, we have the long exact sequence

$$\cdots \to \pi_{n+1}(B) \to \pi_n(X) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(X) \to \cdots$$

3. $\pi_*(X) = \pi_{*-1}(\Omega X)$

Mod p homotopy decomposition

Unfortunately, we can hardly expect a homotopy equivalence

$$X \simeq X_1 \times X_2 \times \cdots \times X_n.$$

So just as in algebra, we consider "one prime at a time" by taking the *localisation* $X_{(p)}$ for a prime p and look for a decomposition of it.
The localisation of a space X corresponds to the algebraic localisation of the groups $\pi_*(X)$ and. In fact, for simply connected spaces,

$$\pi_*(X_{(p)}) \simeq \pi_*(X) \otimes \mathbb{Z}_{(p)}$$

$$(\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ is prime to } p \right\})$$

Based loop space

It is still difficult to decompose $X_{(p)}$ into a product. We then take the based loop space $\Omega X_{(p)}$
(the space of based maps from S^1 to $X_{(p)}$)
and consider a decomposition of $\Omega X_{(p)}$:

$$\Omega X_{(p)} \simeq (Y_1)_{(p)} \times \cdots \times (Y_n)_{(p)}.$$

(We often write $\Omega X \simeq_p Y_1 \times \cdots Y_n$)
The based loop functor does not change homotopy groups:

$$\pi_*(\Omega X_{(p)}) = \pi_{*+1}(X_{(p)}) = \pi_{*+1}(X) \otimes \mathbb{Z}_{(p)}.$$

Therefore, if we have $\Omega X \simeq_p Y_1 \times \cdots Y_n$, we can compute

$$\pi_*(X) \otimes \mathbb{Z}_{(p)} = (\pi_{*-1}(Y_1) \oplus \cdots \oplus \pi_{*-1}(Y_n)) \otimes \mathbb{Z}_{(p)}$$

Homogeneous space

Let G be a compact, connected Lie group and H be its closed subgroup. The quotient G/H is called the *homogeneous space*: For example,

- Lie groups are homogeneous spaces (the case when H is trivial)
- $SO(n)/SO(n-1) = S^{n-1}$ (sphere)
- $U(n)/(U(1) \times U(n-1)) = \mathbb{C}P^n$ (complex projective space)
- $O(n+m)/(O(n) \times O(m)) = Gr_n(\mathbb{R}^{n+m})$ (real grassmannian)
- When $H = P$ is a *parabolic subgroup*, G/P is a projective variety called the *flag manifold*
- When H is an involution fixed subgroup of G, G/H is called the *symmetric space*

Note that there is a fibration $\Omega G/H \to H \to G$.

Flag manifold

Let $G_{\mathbb{C}}$ be a simple complex Lie group and B its maximal solvable subgroup (called the Borel subgroup). Then a subgroup $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ is called the parabolic subgroup if it contains B.

The homogeneous space $G_{\mathbb{C}}/P_{\mathbb{C}}$ is known to admit the structure of a projective variety and called the *flag manifold* associated to $P_{\mathbb{C}} \subset G_{\mathbb{C}}$.

Let $G \subset G_{\mathbb{C}}$ be the maximal compact subgroup. Then by Iwasawa-decomposition, we have a diffeomorphism: $G/P \to G_{\mathbb{C}}/P_{\mathbb{C}}$, where $P := G \cap P_{\mathbb{C}}$. Note that if G is not simply-connected, we can replace it with its universal cover without changing the quotient G/P.

Importance of flag manifolds

Flag manifolds have rich structures and appear in various fields:

- algebraic geometry: projective variety
- symplectic geometry: Hamiltonian torus action
- algebraic topology: homogeneous
- combinatorics: Schubert calculus
- representation theory: Borel-Weil, Springer theory

Thus, a theorem can be interpreted in many ways. For example, Littlewood-Richardson's theorem computes

- the ring structure of grassmannian manifolds
- irreducible decomposition of tensor representations of the symmetric groups
- the number of the solutions to certain geometric problems

Our goal

Problem

Compute $\pi_*(G/P) \otimes \mathbb{Z}_{(p)}$ for a prime p.

We accomplish this goal by giving a mod p homotopy decomposition for $\Omega G/P_{(p)}$.
Then, the problem is reduced to the computation of the homotopy groups of spheres, which is one of the fundamental open problems in homotopy theory.

Our method works only for larger primes
(more precisely, the quasi regular primes of G).

Here's a list of some famous mod p decompositions of (the based loop spaces of) homogeneous spaces:

- ([Serre]) For $p > 2$, $\Omega S^{2n} \simeq_p S^{2n-1} \times \Omega S^{4n-1}$.
- ([Mimura-Nishida-Toda]) For a compact, simple, simply-connected Lie group G, there is a prime p' such that

  $$G \simeq_p B_1 \times \cdots \times B_n \quad (p \geq p'),$$

  where each factor B_i is either an odd sphere S^{2k-1} or the *mod p Stiefel manifold* $B(2k-1, 2k+2p-3)$ defined by the pullback:

  $$\begin{array}{ccccc} S^{2n-1} & \hookrightarrow & B(2n-1, 2n+2p-3) & \twoheadrightarrow & S^{2n+2p-3} \\ \| & & \downarrow & & \downarrow \tfrac{1}{2}\alpha_1(2n) \\ S^{2n-1} & \longrightarrow & O(2n+1)/O(2n-1) & \longrightarrow & S^{2n}. \end{array}$$

  The primes $p \geq p'$ are called the *quasi-regular primes for G*.

We use/extend these results.

First observation

Proposition

When $H \hookrightarrow G$ is null-homotopic, we have

$$\Omega G/H \simeq \Omega G \times H.$$

Proof.

In the fibration sequence $\Omega G \xrightarrow{s} \Omega G/H \xrightarrow{t} H \to G \to G/H$, t has a left inverse t^{-1} by assumption. We have the homotopy equivalence

$$\mu(s \times t^{-1}) : \Omega G \times H \to \Omega G/H,$$

where μ is the loop multiplication in $\Omega G/H$. $\square$

First observation

When G is simply-connected and $H = T$ is its maximal torus, G/T is called the *complete flag manifold*. Since $\pi_1(G) = 0$, the inclusion $T \to G$ is null-homotopic. By the previous Proposition, we have

Corollary

$$\Omega G/T \simeq \Omega G \times T.$$

In particular, (since $\pi_2(G) = 0$)

$$\pi_n(G/T) = \begin{cases} 0 & (n = 1) \\ \mathbb{Z}^{\dim(T)} & (n = 2) \\ \pi_n(G) & (n > 2) \end{cases}$$

Homotopy groups of G

Theorem (Mimura-Nishida-Toda)

For a quasi-regular prime p

$$G \simeq_p B_1 \times \cdots \times B_n,$$

where each factor B_i is either an odd sphere S^{2k-1} or $B(2k-1, 2k+2p-3)$.

Example

$$SU(n) \simeq_p \prod_{i=2}^{n-p+1} B(2i-1, 2i+2p-3) \times \prod_{j=\max(2,n-p+2)}^{\min(n,p)} S^{2j-1} \quad (p > n/2)$$

$$SO(2n+1) \simeq_p \prod_{i=1}^{n-\frac{p-1}{2}} B(4i-1, 4i+2p-3) \times \prod_{j=n-\frac{p-3}{2}}^{\min(n,\frac{p-1}{2})} S^{4j-1} \quad (p > n)$$

Theorem (Toda)

$$\pi_{2m-1+t}(S^{2m-1}_{(p)}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-1, 1 \le i \le p-1 \\ \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-2, m \le i \le p-1 \\ 0 & \text{otherwise for } 1 \le t \le 2p(p-1)-3 \end{cases}$$

Theorem (Mimura-Toda,Kishimoto)

$$\pi_{3+t}(B(3,2p+1)_{(p)}) = \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & t = 2i(p-1)-1, 2 \le i \le p-1 \\ \mathbb{Z}_{(p)} & t = 2p-2 \\ 0 & \text{otherwise for } 1 \le t \le 2p(p-1)-3 \end{cases}$$

$$\pi_{2m-1+t}(B(2m-1,2m+2p-3)_{(p)}) = \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & t = 2i(p-1)-1, 2 \le i \le p-1 \\ \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-2, m \le i \le p-1 \\ \mathbb{Z}_{(p)} & t = 2p-2 \\ 0 & \text{otherwise for } 1 \le t \le 2p(p-1)-3 \end{cases}$$

Key machinery

To tackle the general case when $H \hookrightarrow G$ is not null-homotopic, we use the following results on finite H-spaces:

Theorem (Cohen-Neisendorfer)

Let $\mathcal{C}_p$ be the category of CW-complexes consisting of ℓ odd dimensional cells, where $\ell < p-1$. Let $\mathcal{H}$ be the category of finite H-spaces. Then there is a functor $M : \mathcal{C}_p \longrightarrow \mathcal{H}$ which satisfies

- there is an isomorphism of Hopf algebras $H_*(M(A)) \cong \Lambda(\widetilde{H}_*(A))$;
- a cofibration $A_1 \longrightarrow A_2 \longrightarrow A_3$ will be converted to a fibration $M(A_1) \longrightarrow M(A_2) \longrightarrow M(A_3)$.

The functor M

- The exterior algebra functor $\bigwedge$ associates a ring $\bigwedge(V)$ to a free module V.
- $V \hookrightarrow \bigwedge(V)$ is the inclusion of the generators
- Any ring homomorphism $f : \bigwedge(V) \to R$ is determined by its restriction on the generators $f : V \to R$

M is a homotopy theoretical analogy of $\bigwedge$:

- The functor M associates an H-space $M(A)$ to a space A.
- $H_*(A) \hookrightarrow H_*(M(A)) = \bigwedge(\widetilde{H}_*(A))$ is the inclusion of the generators
- Any map $f : M(A) \to G$ which respects the product is determined by its restriction on the "generators" $f : A \to G$

From now on, p is always assumed to be a quasi-regular prime of G. It is well-known $H_*(G_{(p)}) = \bigwedge(x_1, x_2, \ldots, x_r)$ as algebras.

Corollary

There is a subspace $A = \bigvee_{i=1}^{t} A_i \subset G$ such that $\mathrm{rank}(\tilde{H}_*(A_i)) \leq 2$ for all $1 \leq i \leq t$ and $M(A) \simeq_p G$.

That is, A is the homotopy theoretical "generators" of the group G. Furthermore, each factor has only up to two generators in the homology.

Example

Let $G = SU(4)$ and $p = 3$. $H_*(SU(4)) = \bigwedge(x_3, x_5, x_7)$. We have

$$A_1 \vee A_2 \hookrightarrow SU(4), \quad (\tilde{H}_*(A_1) = \langle x_3, x_7 \rangle, A_2 = S^5)$$

Applying M on the left gives

$$B(3,7) \times S^5 \simeq_p SU(4).$$

Main result

Theorem (K-Theriault-Ohsita)

Suppose that there is a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_{i=1}^{t} A'_i & \xrightarrow{\bigvee_{i=1}^{t} q_i} & \bigvee_{i=1}^{t} A_i \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ H & \longrightarrow & G \end{array}$$

where $A'_i, A_i \in \mathcal{C}_p$ for $1 \leq i \leq t$, there are Hopf algebra isomorphisms $H_*(H) \cong \Lambda(\widetilde{H}_*(\bigvee_{i=1}^{t} A'_i))$ and $H_*(G) \cong \Lambda(\widetilde{H}_*(\bigvee_{i=1}^{t} A_i))$, and j', j induce the inclusions of the generating sets in homology. Then, there is a homotopy equivalence

$$\Omega(G/H) \simeq_p \prod_{i=1}^{t} \mathrm{fib}(M(q_i))$$

Main result

The homotopy type of

$$\mathrm{fib}(M(q_i)) \hookrightarrow M(A_i') \xrightarrow{M(q_i)} M(A_i)$$

can be determined by the restriction on the generators of

$$H_*(H) \to H_*(G).$$

Note that $H_*(BG) \simeq H_*(BT)^{W(G)}, H_*(BH) \simeq H_*(BT)^{W(H)}$, where $W(G)$ and $W(H)$ are the Weyl groups. Therefore, the above map can be determined by computing the Weyl group invariants.

Now, we analyse each pair of $P \subset G$ by using its classification.

Classification of flag manifold

Flag manifolds G/P are classified by the subsets of the Dynkin diagram of G.

Example

$$G = SU(6) : \circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ$$

$$P_1 : \circ\!-\!\circ \quad\quad \circ\!-\!\circ$$

$$P_2 : \quad \circ \quad\quad \circ\!-\!\circ$$

$$P_3 : \circ\!-\!\circ \quad\quad\quad \circ$$

Then,

$$G/P_1 = \frac{SU(6)}{S(U(3) \times U(3))} = Gr_3(\mathbb{C}^6)$$

$$G/P_2 = G/P_3 = \frac{SU(6)}{S(U(2) \times U(3) \times U(1))}$$

Examples

Let $G = SU(4), P = S(U(2) \times U(2))$ and $p = 3$.
$G/P = Gr_2(\mathbb{C}^4)$ is the Grassmaniann manifold of two planes in $\mathbb{C}^4$. We have

$$\begin{array}{ccc} S^1 \vee S^3 \vee S^3 & \xrightarrow{q} & (e^3 \cup e^7) \vee S^5 \\ \downarrow{j'} & & \downarrow{j} \\ S(U(2) \times U(2)) & \longrightarrow & SU(4). \end{array}$$

To identify q, we look at

$$\begin{array}{ccc} H_*(S(U(2) \times U(2))) & \longrightarrow & H_*(SU(4)) \\ \| & & \| \\ \bigwedge(y_1, y_3, y_3') & \longrightarrow & \bigwedge(x_3, x_5, x_7) \end{array} .$$

We can show (by computing the Weyl group invariants)

$$\bigwedge(y_1, y_3, y_3') \to \bigwedge(x_3, x_5, x_7)$$

is non-trivial only in degrees zero and three.
Therefore, we have

$$q \sim (q_1 \vee q_2 \vee q_3) : S^1 \vee S^3 \vee S^3 \to (e^3 \cup e^7) \vee S^5,$$

where $q_1 : S^1 \to *, q_2 : S^3 \hookrightarrow (e^3 \cup e^7), q_3 : S^3 \xrightarrow{\text{trivial}} S^5$ and

$$\mathrm{fib}(M(q_1)) = S^1, \mathrm{fib}(M(q_2)) = \Omega S^7, \mathrm{fib}(M(q_2)) = S^3 \times \Omega S^5.$$

To summarize, $\Omega Gr_2(\mathbb{C}^4) \simeq_p S^1 \times \Omega S^7 \times S^3 \times \Omega S^5$.
We can use the results on $\pi_*(S^n) \otimes \mathbb{Z}_{(p)}$ to compute $\pi_*(Gr_2(\mathbb{C}^4)) \otimes \mathbb{Z}_{(p)}$.

Examples

Here is a list of decompositions of some flag manifolds:

$$\Omega Gr_m(\mathbb{C}^n) \simeq_p \prod_{j=1}^{m} S^{2j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{2j-1} \quad (p > n/2, 2m \leq n)$$

$$\Omega \frac{G_2}{SO(4)} \simeq_p S^3 \times \Omega S^{11} \quad (p \geq 5)$$

$$\Omega \frac{F_4}{SU(2) \cdot Sp(3)} \simeq_p \begin{cases} S^3 \times S^7 \times \Omega B(15, 23) & (p = 5) \\ S^3 \times S^7 \times \Omega S^{15} \times \Omega S^{23} & (p \geq 7) \end{cases}$$

$$\Omega \frac{E_6}{T^1 \cdot Spin(10)} \simeq_p S^1 \times S^7 \times \Omega S^{17} \times \Omega S^{23} \quad (p \geq 5)$$

$$\Omega \frac{E_7}{S^1 \cdot E_6} \simeq_p S^1 \times S^9 \times S^{17} \times \Omega S^{19} \times \Omega S^{27} \times \Omega S^{35} \quad (p \geq 11)$$

Homotopy exponent

For a quasi-regular prime p for G, from the decomposition of $\Omega G/P$:

- We can compute the homotopy groups $\pi_*(G/P) \otimes \mathbb{Z}_{(p)}$ up to a certain degree
  (up to the knowledge of the homotopy groups of S^n and $B(2n-1, 2n-2p-3)$)
- We can give bounds (in fact, explicit values in many cases) for the p-primary homotopy exponents of G/P.

Definition
The *p-primary homotopy exponent* $\exp_p(X)$ of a space X is the least power of p that annihilates the p-torsion in $\pi_*(X)$.

Homotopy exponent

For the computation of homotopy exponents, we use the following results:

Theorem (Cohen-Moore-Neisendorfer)

Let $p \geq 5$. Then $\exp_p(S^{2n+1}) = p^n$

Theorem (Davis-Theriault)

Let $p \geq 5$. Then $\exp_p(B(3, 2p+1)) = p^{p+1}$ and for $k > 2$,

$$p^{k+p-2} \leq \exp_p(B(2k-1, 2k+2p-3)) \leq p^{k+p-1}.$$

Examples

Here is a list of the *p*-primary exponents of some flag manifolds:

$$\exp_p(Gr_m(\mathbb{C}^n)) = p^{2n-1} \quad (p > n/2, 2m \leq n)$$
$$\exp_p\left(\frac{G_2}{SO(4)}\right) = p^5 \quad (p \geq 5)$$
$$\exp_p\left(\frac{F_4}{SU(2)\cdot Sp(3)}\right) = p^{11} \quad (p \geq 7)$$
$$\exp_p\left(\frac{E_6}{T^1\cdot Spin(10)}\right) = p^{11} \quad (p \geq 5)$$
$$\exp_p\left(\frac{E_7}{S^1\cdot E_6}\right) = p^{17} \quad (p \geq 11)$$

Conclusion

1. We gave a general method to give a mod p decomposition of the based loop space $\Omega G/H$ of a homogeneous space G/H when p is a quasi-regular prime of G.
2. The decomposition allows us to compute (the localisation of) the homotopy groups of G/H and, in particular, the p-primary exponents of G/H.
3. In practice, we have to compute the induced map $H^*(BG) \to H^*(BH)$ to use the above method.
4. For flag manifolds, by using the classification based on the Dynkin diagram, we can carry out concrete computations.

Future work

- Extend the decomposition result to smaller primes
- Can we avoid case by case analysis based on the classification?
- Can we find a relation with Schubert calculus?

Thank you very much