Torus equivariant cohomology of flag manifolds

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Outline

- Introduction: Cohomology of Grassmann manifolds
- Cohomology of flag manifolds
  - Chevalley presentation
  - Borel presentation
- Torus action and the equivariant cohomology
- Three presentations of the equivariant cohomology of flag manifolds
  - Chevalley presentation
  - Borel presentation
  - GKM presentation

Enumerative geometry

- How many lines are there on the plane which go through given two points?
- How many points are there which lie both on a line and a degree n curve on the (complex) plane?
- How many points are there which lie both on a degree m and a degree n curves on the (complex projective) plane?
- How many lines are there which intersect given four lines in the three dimensional (complex projective) space.
- How many lines are there on a cubic surface?

Last two questions are solved using the cohomology of a Grassmann manifold!

Let $Gr_m(\mathbb{C}^n)$ be the Grassmanniann manifold

$$Gr_m(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \mid \dim(V) = m\}.$$

Theorem (Ehresmann)

The ordinary cohomology $H^*(Gr_m(\mathbb{C}^n); \mathbb{Z})$ (which is isomorphic to the Chow ring $A^*(Gr_m(\mathbb{C}^n))$) is a free module generated by the Schubert classes $\{X_\lambda\}$ indexed by *Young diagrams* $\lambda = (n - m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ contained in the $m \times (n - m)$ rectangle.

The Schubert class is represented by the Schubert variety:

$$X_\lambda := \{V \in Gr_m(\mathbb{C}^n) \mid \dim(V \cap \mathbb{C}^{m+i-\lambda_i}) \geq i, \ (1 \leq \forall i \leq m)\}$$

The codimension of X_λ is $2\sum_{i=1}^m \lambda_i$.

Regard $Gr_2(\mathbb{C}^4)$ as the set of lines in $\mathbb{C}P^3$.
Then, X_λ is the set of lines which satisfies a certain condition:

- $X_{\emptyset}$: without any restriction
- $X_{\ydiagram{1}}$: intersecting a given line
- $X_{\ydiagram{2}}$: going through a given point
- $X_{\ydiagram{1,1}}$: lying on a given plane
- $X_{\ydiagram{2,1}}$: lying on a given plane and going through a given point on the plane
- $X_{\ydiagram{2,2}}$: lying on a given line, i.e. the line itself

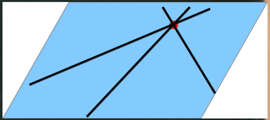


Figure: $X_{\ydiagram{2,1}}$

For example, $X_{\square}$ is the sub-variety consisting of "lines in $\mathbb{C}P^3$ which intersect a given line."
Then, the intersection

$$X_{\square} \cap X_{\square} \cap X_{\square} \cap X_{\square}$$

is the sub-variety consisting of lines which intersect four given lines.
We want to compute it to answer the following question:

"How many lines are there which intersect all four given lines in $\mathbb{C}P^3$?"

Since $H^*(Gr_m(\mathbb{C}^n);\mathbb{Z})$ is a free module over X_λ's, any element is written as a linear combination of them. In particular, there are integers c_λ satisfying

$$(X_{\square})^4 = \sum_\lambda c_\lambda X_\lambda.$$

But how can we determine the coefficients c_λ?

Those coefficients are called the *structure constants*.

Borel presentation

One way to answer the question is to use another presentation of $H^*(Gr_m(\mathbb{C}^n);\mathbb{Z})$:

Theorem

$$H^*(Gr_m(\mathbb{C}^n);\mathbb{Z}) = \frac{\mathbb{Z}[x_1,\ldots,x_n]^{S_m\times S_{n-m}}}{\mathbb{Z}^+[x_1,\ldots,x_n]^{S_n}},$$

where S_k is the symmetric (or permutation) group on k letters.

Here, the elements are represented by (invariant) polynomials and the multiplication among them is easily computed.

Borel presentation

For $Gr_2(\mathbb{C}^4)$,
S_4 acts on $\mathbb{Z}[x_1, x_2, x_3, x_4]$ by permuting x_i.
Therefore,

$$H^*(Gr_2(\mathbb{C}^4)) = \frac{\mathbb{Z}[x_1, x_2, x_3, x_4]^{S_2 \times S_2}}{\mathbb{Z}^+[x_1, x_2, x_3, x_4]^{S_4}} = \frac{\mathbb{Z}[c_1, c_2, c_1', c_2']}{(e_1, e_2, e_3, e_4)},$$

where $c_1 = x_1 + x_2, c_2 = x_1 x_2, c_1' = x_3 + x_4, c_2' = x_3 x_4$
(that is, the elementary symmetric function on the first and the last two variables)
and e_i is the i-th elementary symmetric function on $x_1, \ldots, x_4$.

To use Borel presentation to compute products in Schubert basis X_λ, we have to answer the following:

Problem

What is the relation between them? More precisely,

1. Find a polynomial $\sigma_\lambda \in \mathbb{Z}[c_1, c_2, c_1', c_2']$ which represents X_λ
2. Write a polynomial $f \in \mathbb{Z}[c_1, c_2, c_1', c_2']$ as a linear combination of X_λ's

1. X_λ is represented by the *Shur polynomial* s_λ
2. For a polynomial $f \in \mathbb{Z}[c_1, c_2, c_1', c_2']$, the coefficients in

$$f = \sum_\lambda c_\lambda X_\lambda$$

   is computed by Newton's *divided difference operators*.

These will be introduced later in a general setting. For the time being, let's look at an example:

1. $\sigma_{\square} = c_1$ and $\sigma_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} = c_2^2$
2. $c_1^4 \equiv 2c_2^2$ modulo the ideal. So we have

$$(X_{\square})^4 = c_1^4 \equiv 2c_2^2 = 2X_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}.$$

Lines on a cubic surface

Let's give the answer to "How many lines are there on a cubic surface?"

Let $\gamma \to Gr_2(\mathbb{C}^4)$ be the canonical bundle.
What we want to compute is the zero locus of a generic section of the symmetric tensor $S^3(\gamma)$, which is represented by the top chern class of $S^3(\gamma)$.

$$S^3(\gamma) = \langle x_1 \otimes x_1 \otimes x_1, x_1 \otimes x_1 \otimes x_2, x_1 \otimes x_2 \otimes x_2, x_2 \otimes x_2 \otimes x_2 \rangle$$

and hence, the total chern class is

$$\begin{aligned} c(S^3(\gamma)) &= (1+3x_1)(1+2x_1+x_2)(1+x_1+2x_2)(1+3x_2) \\ &= 1 + 6c_1 + (11c_1^2 + 10c_2) + (6c_1^3 + 30c_1c_2) + (18c_1^2c_2 + 9c_2^2). \end{aligned}$$

We have the answer 27 since $18c_1^2c_2 + 9c_2^2 = 27X_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}$.

Mysterious connection

Irreducible representations V_λ of symmetric groups are also indexed by the Young diagrams.
The character of V_λ is given by the Schur function s_λ. The consequence is, the coefficients in

$$V_\lambda \otimes V_\mu = \sum_\tau c_\tau V_\tau$$

is SAME as the ones in

$$X_\lambda \cap X_\mu = \sum_\tau c_\tau X_\tau.$$

We do not know a satisfactory explanation for this mysterious correspondence between representation theory and topology.

General setting

notation	typical example
G: simple complex Lie group	$GL_n(\mathbb{C})$
B: Borel subgroup	upper triangular matrices
P: Parabolic subgroup	upper triangular block matrices
T: maximal torus	diagonal matrices
G/P: flag manifold	$Gr_m(\mathbb{C}^n) = GL_n(\mathbb{C})/P_m$
$W(G) = N_G(T)/T$: Weyl group of G	S_n: symmetric group
$W(P) = N_P(T)/T$: Weyl group of P	$S_m \times S_{n-m}$: symmetric group
$\mathfrak{t}^*$: dual Lie algebra of T	$\mathbb{R}^n = \langle t_1, \ldots, t_n \rangle$
Φ: positive roots	$t_i - t_j$ $(i > j)$
$\alpha_1, \ldots, \alpha_{n-1}$: simple roots	$t_{i+1} - t_i$
s_α: reflection by $\alpha \in \Phi$	
$s_1, \ldots, s_{n-1}$: simple reflections	$s_i = (i, i+1)$: simple transposition

G/P has the *Bruhat cell decomposition*

$$G/P = \bigcup_{w \in W(G)/W(P)} Bw_0wP/P,$$

where the cell Bw_0wP/P has codimension $2l(w)$. The Schubert variety X_w is the closure of the cell. Consequently, we have

Theorem (Chevalley)

$H^*(G/P)$ is a free module generated by the Schubert classes $\{X_w\}$ indexed by the coset $W(G)/W(P)$.

Note that $H^*(G/P) \simeq A^*(G/P)$.

The problem is to compute the structure constants c_w in

$$X_u \cap X_v = \sum_{w \in W(G)/W(P)} c_w X_w.$$

This is a modern interpretation of Hilbert's 15th problem.

From now on, we assume that the coefficients for the cohomology is taken to be $\mathbb{Q}$.
The Eilenberg-Moore spectral sequence for the fiber bundle

$$G/P \hookrightarrow BP \to BG$$

collapses at E_2-term. This together with the fact that $H^*(BP) = H^*(BT)^{W(P)}$ and $H^*(BG) = H^*(BT)^{W(G)}$, we have

Theorem (Borel)

$$H^*(G/P) = \frac{H^*(BT)^{W(P)}}{(H^+(BT)^{W(G)})}.$$

Let $n = \dim(T)$. $BT = (\mathbb{C}P^\infty)^n$ and $H^*(BT) = \mathbb{Q}[x_1, \ldots, x_n]$ is the polynomial algebra and the Weyl groups $W(G)$ and $W(P)$ act on it by the standard representation.

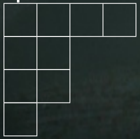
Young diagram $\Leftrightarrow$ coset

$\lambda = (n - m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ corresponds to the *Grassmann permutation* $W(GL_n(\mathbb{C}))/W(P_m) = S_n/S_m \times S_{n-m}$

$$\begin{pmatrix} 1 & 2 & \cdots & m-n-1 & m-n & \cdots & n \\ \lambda_m + 1 & \lambda_{m-1} + 2 & \cdots & \lambda_1 + m & j_{m-n} & \cdots & j_n \end{pmatrix},$$

where $j_{m-n} < \ldots < j_n$ are those numbers which do not appear as $\lambda_i + m - i + 1$.

Example



$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 5 & 8 & 1 & 3 & 6 & 7 \end{pmatrix} \in S_8/S_4 \times S_4$$

Remark on $\mathbb{Z}$-coeffs

Chevalley presentation is valid with $\mathbb{Z}$-coefficients:

$$H^*(G/P) = \mathbb{Z}\langle X_w \mid w \in W(G)/W(P)\rangle$$

While Borel presentation works only with a ring such that the torsion primes of $H_*(G;\mathbb{Z})$ are invertible. (Borel-type presentations with $\mathbb{Z}$-coefficients have been computed in these 40 years by case by case analysis by Toda-Watanabe, Nakagawa, K-Nakagawa, Duan-Zhao)

Nevertheless, since $H^*(G/P;\mathbb{Z}) \to H^*(G/P;\mathbb{Q})$ is injective, the method we will describe in the following is valid to determine the structure constants with $\mathbb{Z}$-coefficients.

Problem

What is the relation between them? More precisely,

1. Find a polynomial $\sigma_w \in H^*(BT)^{W(P)}$ which represents X_w
2. Write a polynomial $f \in H^*(BT)^{W(P)}$ as a linear combination of X_w's

σ_w is called the *Schubert polynomial*. Note that it has indeterminacy up to the ideal $H^+(BT)^{W(G)}$.

The answer is given by the divided difference operators introduced by Berstein-Gelfand-Gelfand and Demazure.

Divided difference operator

For a simple reflection $s_i \in W(G)$, define the divided difference operator $\Delta_{s_i} : \mathbb{Q}[x_1, \ldots, x_n] \to \mathbb{Q}[x_1, \ldots, x_n]$ by

$$\Delta_{s_i}(f(x)) = \frac{f(x) - s_i f(x)}{-\alpha_i(x)}$$

Theorem (BGG)

For the longest element $w_0 \in W(G)$,

$$\sigma_{w_0} = \frac{\prod_{\alpha \in \Phi}(-\alpha)}{|W|}$$

Then, recursively

$$\sigma_w = \Delta_{s_i} \sigma_{ws_i} \quad (l(ws_i) = l(w) + 1)$$

Divided difference operator

For a simple root α_i, let P_i be the minimal parabolic subgroup with the Lie algebra $L(P_i) = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_i}$.
Then, we have $P_i/B \simeq \mathbb{C}P^1$ and we have the following bundle

$$\mathbb{C}P^1 \hookrightarrow G/B \xrightarrow{\pi} G/P_i$$

Topologically, the operator Δ_{s_i} is given by the composition

$$\pi^* \circ \pi_* : H^*(G/B) \to H^{*-2}(G/B),$$

where $\pi_* : H^*(G/B) \to H^{*-2}(G/P_i)$ is the Gysin map.

(Chevalley) $\Rightarrow$ (Borel)

Example

- Let $G = GL_3(\mathbb{C})$ and $P = B$.
- $W(G) = S_3 = \langle s_1, s_2 \rangle$ and $w_0 = s_1 s_2 s_1$
- $\sigma_{w_0} = \frac{1}{6}(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$
-

$$\begin{aligned}\sigma_{s_1} &= \Delta_{s_2}(\sigma_{s_1 s_2}) = \Delta_{s_2} \circ \Delta_{s_1}(\sigma_{s_1 s_2 s_1}) \\ &= \Delta_{s_2} \circ \Delta_{s_1} \left(\frac{1}{6}(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) \right) \\ &= \Delta_{s_2} \left(\frac{1}{3}(x_2 - x_3)(x_1 - x_3) \right) = \frac{1}{3}(2x_1 - x_2 - x_3)\end{aligned}$$

(Borel) $\Rightarrow$ (Chevalley)

For $w \in W(G)$, let $\Delta_{s_{i_1} s_{i_2} \cdots s_{i_l}} := \Delta_{s_{i_1}} \circ \cdots \circ \Delta_{s_{i_l}}$.
$f(x) \in H^*(BT)^{W(P)} = \mathbb{Q}[x_1, \ldots, x_n]^{W(P)}$ is equal to

$$\sum_{w \in W(G)/W(P)} \Delta_w(f) X_w,$$

where the sum runs for all w with $l(w) = \deg(f)$.

Example

- Let $G = GL_3(\mathbb{C})$ and $P = B$.
- Let $f = x_1 x_2 + x_3^2 \in H^*(BT)$
- $f \simeq \Delta_{s_1 s_2}(f) X_{s_1 s_2} + \Delta_{s_2 s_1}(f) X_{s_2 s_1} = 2 X_{s_1 s_2}$

Equivariant generalisation

A flag manifold G/P admits the action of the maximal torus T by the left multiplication.
The equivariant cohomology $H_T^*(G/P)$ is defined to be the ordinary cohomology of the Borel construction

$$H_T^*(G/P) := H^*(ET \times_T G/P),$$

where $T \hookrightarrow ET \to BT$ is the universal T-bundle.
$H_T^*(G/P)$ is an algebra over $H^*(BT)$ through the projection $ET \times G/P \to BT$ and

$$H_T^*(G/P) \otimes_{H^*(BT)} \mathbb{Q} = H^*(G/P).$$

That is, if we now $H_T^*(G/P)$, we can recover $H^*(G/P)$ easily.

$H_*^T(G/P)$ is a more complicated object which contains $H^*(G/P)$ as the "constant part."

Strangely, we have a new presentation for $H_T^*(G/P)$ which is not available for $H^*(G/P)$.

There are three presentations:

- (Chevalley) $H_T^*(G/P)$ is a free $H^*(BT)$-module generated by $\{X_w\}$.
- (Borel) $H_T^*(G/P) = H^*(BT) \otimes_{H^*(BT)^{W(G)}} H^*(BT)^{W(P)}$
- (Goresky-Kottwitz-MacPherson, Kostant-Kumar) *GKM presentation*

Chevalley presentation

Schubert varieties X_w are T-stable, and thus, define classes in $H_T^*(G/P)$.

They generate $H_T^*(G/P)$ freely over $H^*(BT) = \mathbb{Q}[t_1, \dots, t_n]$:

$$H_T^*(G/P) = H^*(BT) \otimes \langle X_w \mid w \in W(G)/W(P) \rangle,$$

That is, any element $z \in H_T^*(G/P)$ is written as a linear sum

$$z = \sum_{w \in W(G)/W(P)} f_w(t) X_w \qquad (f(t) \in H^*(BT)).$$

Borel presentation

$$H_T^*(G/P) = H^*(BT) \otimes_{H^*(BT)^{W(G)}} H^*(BT)^{W(P)}$$

- For left factor, denote $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$
- For right factor, denote $H^*(BT)^{W(P)} = \mathbb{Q}[x_1, \ldots, x_n]^{W(P)}$
- Any element is represented by a "double polynomial" $f(t; x) \in \mathbb{Q}[t_1, \ldots, t_n] \otimes \mathbb{Q}[x_1, \ldots, x_n]^{W(P)}$

For example,

$$H_T^*(Gr_2(\mathbb{C}^4)) = \frac{\mathbb{Q}[t_1, \ldots, t_4] \otimes \mathbb{Q}[c_1, c_2, c_1', c_2']}{(e_1(t) - e_1(x), e_2(t) - e_2(x), e_3(t) - e_3(x), e_4(t) - e_4(x))},$$

where $c_1 = x_1 + x_2, c_2 = x_1 x_2, c_1' = x_3 + x_4, c_2' = x_3 x_4$.

GKM presentation

$$H_T^*(G/P) = \{h_w \in \bigoplus_{w\in W(G)/W(P)} H^*(BT) \mid h_w - h_v \in (\alpha) \text{ if } w = s_\alpha v\}$$

- It is derived by the *localization to the fixed points map*

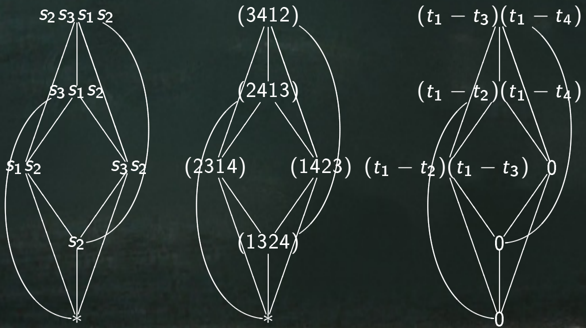
$$H_T^*(G/P) \to \bigoplus_{w\in W(G)/W(P)} H_T^*(wP/P) = \bigoplus_{w\in W(G)/W(P)} H_T^*(BT)$$

- We can draw a graph (called the GKM-graph) with vertices $W(G)/W(P)$ and edges with labels $\{ \, v \overset{\alpha}{\text{———}} w \mid w = s_\alpha v\}$
- An element is represented by a series of polynomials $\{h_w\}$ indexed by the vertices
- $h_w - h_v$ should be divisible by α if v and w are connected by an edge with the label α

GKM presentation

Example

Let $G = GL_4(\mathbb{C})$ and $P = P_2$ so that $G/P = Gr_2(\mathbb{C}^4)$.
The GKM-graph and an element in $H_T^*(G/P)$ are given by



How these three presentations are related?

(Borel) $\Rightarrow$ (Chevalley)

$f(t;x) \in H^*(BT) \otimes H^*(BT)^{W(P)} = \mathbb{Q}[t_1, \ldots, t_n] \otimes \mathbb{Q}[x_1, \ldots, x_n]^{W(P)}$ is equal to

$$\sum_{w \in W(G)/W(P)} \Delta_w(f)(t;t) X_w.$$

That is, first apply Δ_w and evaluate at $x_i = t_i \ \forall i$.

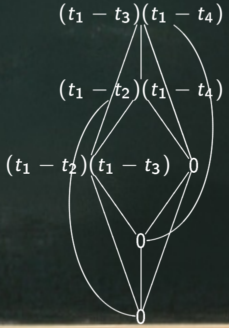
Example

- Let $G = GL_3(\mathbb{C})$ and $P = B$.
- $f = t_1 x_1 x_2 = f(t;t) + \Delta_{s_1}(f)(t;t)X_{s_1} + \Delta_{s_2}(f)(t;t)X_{s_2} + \Delta_{s_1 s_2}(f)(t;t)X_{s_1 s_2} + \Delta_{s_2 s_1}(f)(t;t)X_{s_2 s_1} + \Delta_{s_1 s_2 s_1}(f)(t;t)X_{s_1 s_2 s_1}$ $= t_1^2 t_2 + t_1^2 X_{s_2} + t_1 X_{s_1 s_2}$

(Borel) $\Rightarrow$ (GKM)

Let $f(t;x) \in H^*(BT) \otimes_{H^*(BT)^{W(G)}} H^*(BT)^{W(P)}$.
Then, the corresponding element in the GKM-presentation is given by evaluating at $x_i = w^{-1}(t_i)$: $h_w(t) = f(t; w^{-1}(t))$.

Example
For $G/P = Gr_2(\mathbb{C}^4)$ and $f(t;x) = (x_2 - t_1)(x_1 - t_1)$,



(GKM) ⇒ (Chevalley)

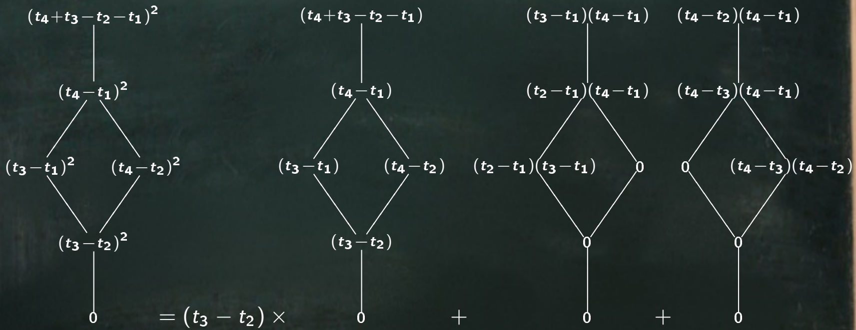
Let $G_w := \{g_v | v \in W\}$ be the element in the GKM presentation which corresponds to X_w.
G_w is characterized by the following properties:

- $\deg(g_v) = l(w)$ for all $v \in W$.
- $g_v = 0$ unless $v \geq w$
- $g_w = \prod \alpha$ where α runs all such that $s_\alpha v < w$.

By this "upper triangularity," we can express any element $\{h_v \mid v \in W\}$ in GKM presentation in terms of a linear combination of X_w.

(GKM) $\Rightarrow$ (Chevalley)

For $G/P = Gr_2(\mathbb{C}^2)$,



$$= (t_3 - t_2)X_{s_2} + X_{s_1 s_2} + X_{s_3 s_2}$$

(Chevalley) $\Rightarrow$ (GKM)

Theorem (Billey)

Let $G_w := \{g_v | v \in W\}$ be the element in GKM presentation which corresponds to X_w. Then,

$$g_v(t;x) = \sum \beta_{j_1} \cdots \beta_{j_{l(w)}}$$

where $\beta_k = s_{i_1} \cdots s_{i-1} \alpha_{i_k}$ and the sum runs over $(1 \leq j_1 < \cdots < j_{l(w)} \leq l(v))$ such that $s_{i_{j_1}} \cdots s_{i_{j_{l(w)}}} = w$.

(Chevalley) ⇒ (GKM)

Example

Let $G_{s_1 s_2} = \{g_v\} \in H_T^*(Gr_2(\mathbb{C}^2))$ be the GKM-presentation of $X_{s_1 s_2}$.

$$
\begin{aligned}
g_{s_1 s_2} &= \beta_1 \beta_2 = \alpha_1 s_1(\alpha_2) \\
&= (t_2 - t_1)(t_3 - t_1) \\
g_{s_3 s_1 s_2} &= \beta_2 \beta_3 = s_3(\alpha_1) s_3 s_1(\alpha_2) \\
&= (t_2 - t_1)(t_4 - t_1) \\
g_{s_2 s_3 s_1 s_2} &= \beta_2 \beta_3 = s_2 s_3(\alpha_1) s_2 s_3 s_1(\alpha_2) \\
&= (t_3 - t_1)(t_4 - t_1)
\end{aligned}
$$

(Chevalley) ⇒ (Borel)

Define the set of partitions of a word $w \in W$:
$P_i(w) := \{(w_1, w_2, \ldots, w_i) \mid w_1 w_2 \cdots w_i = w, l(w_1) + \cdots + l(w_i) = l(w)\}$

Theorem (K)

The following is a polynomial representative of $X_w \in H_T^*(G/P)$:

$$\mathfrak{S}_w(t;x) = \sum_{i=1}^{l(w)} \sum_{P_i(w)} (-1)^{i-1} \sigma_{w_1}(t) \sigma_{w_2}(t) \cdots \sigma_{w_{i-1}}(t)(\sigma_{w_i}(t) - \sigma_{w_i}(x)),$$

where $\sigma_w(t) = \Delta_{w^{-1}w_0} \frac{\prod_{\alpha \in \Phi}(-\alpha)}{|W|}$.

We can replace σ_w with any other *ordinary Schubert polynomial*: For example, if we take Lascoux and Schützenberger's Schubert polynomial for $\sigma_w(t)$, we obtain the double Schubert polynomial by the above formula.

(Chevalley) ⇒ (Borel)

Example

For $Gl_3(\mathbb{C})/B$,

$$\sigma_{[121]} = x_1^2 x_2 \qquad \sigma_{[12]} = x_1 x_2$$
$$\sigma_{[21]} = x_1^2 \qquad \sigma_{[1]} = x_1$$
$$\sigma_{[2]} = x_1 + x_2$$

and hence

$$\begin{aligned}
\mathfrak{S}_{[121]}(t;x) =& \sigma_{[121]}(x) - \sigma_{[1]}(t)\sigma_{[21]}(x) - \sigma_{[2]}(t)\sigma_{[12]}(x) \\
&+(\sigma_{[1]}(t)\sigma_{[2]}(t) - \sigma_{[12]}(t))\sigma_{[1]}(x) + (\sigma_{[2]}(t)\sigma_{[1]}(t) - \sigma_{[21]}(t))\sigma_{[2]}(x) \\
&+(-\sigma_{[121]}(t) + \sigma_{[1]}(t)\sigma_{[21]}(t) + \sigma_{[2]}(t)\sigma_{[12]}(t) \\
&+\sigma_{[12]}(t)\sigma_{[1]}(t) + \sigma_{[21]}(t)\sigma_{[2]}(t) \\
&-\sigma_{[2]}(t)\sigma_{[1]}(t)\sigma_{[2]}(t) - \sigma_{[1]}(t)\sigma_{[2]}(t)\sigma_{[1]}(t)) \\
=& (x_1 - t_1)(x_1 - t_2)(x_2 - t_1)
\end{aligned}$$

Ordinary vs Equivariant

We have two ways to get $\mathfrak{S}_w(t;x)$ from $\sigma_w(x)$.

Proposition (K)

- $\sigma_w(x) = \mathfrak{S}_w(0;x)$
- $\sigma_w(x) = \dfrac{(-1)^{l(w)}}{|W|} \sum_{v \in W} \mathfrak{S}_{w^{-1}}(x; v(x))$
  topologically, the RHS corresponds to the Gysin map associated with the bundle $G/B \hookrightarrow ET \times_T G/B \to BT$.

On the other hand, Theorem (K) gives a way to compute $\mathfrak{S}_w(t;x)$ from $\sigma_v(x)$ for $v \le w$.

Conclusion and open problems

- The ordinary cohomology $H^*(G/P;\mathbb{Q})$ has two presentations: Borel and Chevalley
- The torus equivariant cohomology $H_T^*(G/P;\mathbb{Q})$ has three presentations: Borel, Chevalley, and GKM
- Each of them has both advantages and disadvantages
- There is a method to convert an element from one to another

Open problems:

- How to compute the product $\mathfrak{S}_v\mathfrak{S}_w$ efficiently? (Hilbert's 15th problem)
- Is there a more efficient conversion method?
- How to generalise the theory to the case when P is non-parabolic?

Thank you very much

A Maple package to compute (equivariant) cohomology of flag manifolds is available at
http://skaji.org/code/