A type-A Weyl group action on the associated real toric manifold

Shizuo Kaji
Yamaguchi Univ. / JST PRESTO

Joint with Soojin Cho & Suyoung Choi
Ajou Univ.

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Introduction

When a finite group G acts on a space M,
we obtain a G-module $H^*(M)$

We consider

- M: real toric manifold assoc. to a root system
- $G = W$: the corresponding Weyl group
- $H^*(M)$: a reprentation of W

Introduction

When a finite group G acts on a space M,
we obtain a G-module $H^*(M)$
and $K^*(M), KO^*(M), MU^*(M), \ldots$

We consider

- M: real toric manifold assoc. to a root system
- $G = W$: the corresponding Weyl group
- $H^*(M)$: a reprentation of W

Outline of talk

- Real moment angle complex & Real toric space
- Group action on real toric space
- Real toric manifold assoc. to root systems
- Concrete computation for the type-A case

Real moment angle complex
&
Real toric space

Real moment angle complex

K: simplicial complex over $[m] := \{1, 2, \ldots, m\}$

$$\begin{aligned} \mathbb{R}\mathcal{Z}_K &= (\underline{D^1}, \underline{S^0})^K \\ &= \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma\} \end{aligned}$$

is called the *real moment angle complex*.

$\mathbb{F}_2^m$ acts coordinate-wisely on $\mathbb{R}\mathcal{Z}_K$.

Real toric space

$\Lambda\colon \mathbb{F}_2^m \to \mathbb{F}_2^n$: linear map

$M_{K,\Lambda} := \mathbb{R}\mathcal{Z}_K/\ker\Lambda$ is called the *real toric space*

Alternatively, we can start with any subgroup $H \subset \mathbb{F}_2^m$ and consider

$$M_{K,H} := \mathbb{R}\mathcal{Z}_K/H.$$

Example

When K is a polytopal $(n-1)$-sphere and the column vectors $\Lambda(i_1), \ldots, \Lambda(i_\ell)$ are linearly independent in $\mathbb{F}_2^n$ for any $\{i_1, \ldots, i_\ell\} \in K$, $M_{K,\Lambda}$ is nothing but the *small cover*.

Cohomology of Real moment angle complex

Theorem (L.Cai)

Let R_K be the non-commutative DGA

$$\frac{\mathbb{Z}\langle u_1, \ldots, u_m, t_1, \ldots, t_m \rangle}{(u_i^2, u_i u_j + u_j u_i, u_i t_i - u_i, t_i u_i, t_i u_j - u_j t_i, t_i^2 - t_i, t_i t_j - t_j t_i, u_\sigma \ (\sigma \notin K))}$$

where $i \neq j$ and $d(t_i) = u_i, d(u_i) = 0$ for $i = 1, \ldots, m$.

Then, $H^*(\mathbb{R}\mathcal{Z}_K; \mathbb{Z}) \simeq H^*(R_K)$

Stable decomposition of Real moment angle complex

- K_I for $I \subset [m]$ denotes the restriction (full-subcomplex) of K on I.
- We have an additive isomorphism $R_K = \bigoplus_{I \subset [m]} R_{K_I}$, where $R_{K_I} := \bigoplus_{\sigma \in K_I} \mathbb{Z} u_\sigma t_{I \setminus \sigma} \subset R_K$
- and $H^*(R_{K_I}) \cong \widetilde{H}^{*-1}(K_I; \mathbb{Z})$

Cai's result is reminiscent of

Theorem (Bahri-Bendersky-Cohen-Gitler)

There exists a homotopy equivalence

$$\Sigma \mathbb{R}\mathcal{Z}_K \simeq \bigvee_{I \subset [m]} \Sigma^2 K_I$$

Cohomology of Real toric space

Observe that up to 2-torsion

$$H^*(M_{K,\Lambda}) \cong H^*(\mathbb{R}\mathcal{Z}_K)^{\ker \Lambda} \cong H^*(R_K)^{\ker \Lambda}$$

This invariant ring has a nice presentation:

Theorem (Choi-Park, Suciu-Trevisan)
Let R be a coefficient ring where 2 is invertible. Then,

$$H^*(M_{K,\Lambda}; R) \cong \bigoplus_{I \in Row(\Lambda)} \tilde{H}^*(R_{K_I} \otimes R) \cong \bigoplus_{I \in Row(\Lambda)} \tilde{H}^{*-1}(K_I; R),$$

where $Row(\Lambda) = \mathrm{Im}(\Lambda^T) = (\ker \Lambda)^\perp$ (we identify $\mathbb{F}_2^m \leftrightarrow 2^{[m]}$).

Stable decomposition of Real toric space

Choi-Park's result has a space level counterpart:

Theorem (Choi-K-Theriault or S^3)

Localised at an odd prime p or the rationals,

$$\Sigma(M_{K,\Lambda}) \simeq_p \bigvee_{I \in Row(\Lambda)} \Sigma^2 K_I,$$

Example

Let K be the boundary of a triangle and

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then $M_{K,\Lambda} = \mathbb{R}P^2$. Since each ΣK_I is contractible, the theorem does not work for $p = 2$.

Group action on real toric space

Group action on real toric space

- G: a finite group
- $G \curvearrowright K$: permutation of simplices

Then, G acts on $\mathbb{R}\mathcal{Z}_K$.

If furthermore $\ker \Lambda$ is stable:

$$\forall g \in G, \forall h = (h_1, \ldots, h_m) \in \ker \Lambda$$
$$gh = (h_{g^{-1}(1)}, \ldots, h_{g^{-1}(m)}) \in \ker \Lambda$$

Then, G acts on $M_{K,\Lambda}$.

Group action on real toric space

A sufficient and easy to check condition is:

Lemma

If for each $g \in G$

$$g\Lambda = A_g \Lambda$$

for some $(n \times n)$-matrix A_g, then $\ker \Lambda$ is G-stable.

Example

- P: a simple polytope embedded in $\mathbb{R}^n$
- $G \curvearrowright P$: linear symmetry
- $K = (\partial P)^*$: the dual of its boundary
- Λ: face normal vectors

Then, $H^*(M_{K,\Lambda})$ is a G-module

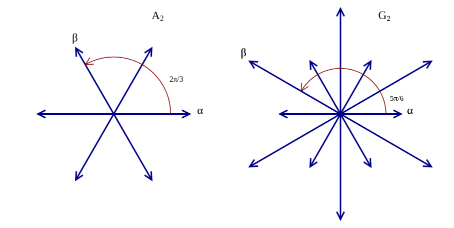
Real toric manifold assoc. to root systems

Root system

A finite set Φ of vectors in $\mathbb{R}^n$ is a root system when

- Φ spans $\mathbb{R}^n$
- For $\alpha \in \Phi$, $c\alpha \in \Phi$ iff $c = \pm 1$.
- For $\alpha \in \Phi$, denote by s_α the reflection through the hyperplane perpendicular to α. Then, $s_\alpha \Phi = \Phi$ for any $\alpha \in \Phi$.
- $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$

Root systems are classified: we have $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$



Root system

For a root system $\Phi \subset \mathbb{R}^n$

- The Weyl group $W := \langle s_\alpha \mid \alpha \in \Phi \rangle$ acts on $\mathbb{R}^n$.
- The exists a basis $\Delta := \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \Phi$ called *simple roots*
- The dual of α_i are called the *fundamental co-weights* denoted by ω_i satisfying $(\omega_i, \alpha_j) = \delta_{ij}$.
- The cone spanned by $\omega_1, \ldots, \omega_n$ is called the *fundamental chamber*
- The *W*-orbit of the fundamental chamber gives a decomposition of $\mathbb{R}^n$ into chambers.

Toric manifold assoc. to root systems

The non-singular complete fan obtained by the chambers gives rise to the *complex toric variety associated to the root system*.

Its cohomology was studied by Procesi, Dolgachev-Lunts, Stembridge, Hiraku Abe, and others.
We are interested in the real version.

Let $K = K_\Phi$ the simplicial complex assoc. with the fan (it is known as the *Coxeter complex*). That is,

- the vertices are rays
- a collection of rays forms a simplex if it spans a wall of a chamber

Let Λ be the column vectors of the coordinates of the rays.
We call $X_\Phi^{\mathbb{R}} = M_{K_\Phi,\Lambda}$ the *real toric manifold assoc. to* Φ

Real toric manifold assoc. to root systems

The p-local homotopy equivalence is W-equivariant:

$$\Sigma(X^{\mathbb{R}}_{\Phi}) \simeq_p \bigvee_{I \in Row(\Lambda)} \Sigma^2 K_I,$$

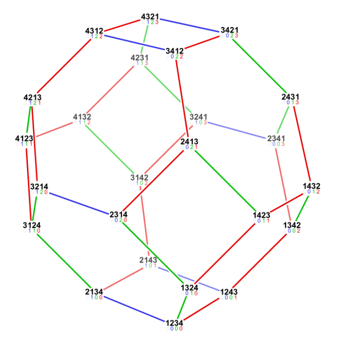
where W acts on $\bigvee_{I \in Row(\Lambda)} \Sigma K_I$ by permuting vertices.

Observe that $K_I \simeq K_{wI}$ for $I \in Row(\Lambda)$ and $w \in W$.

Example

$$\begin{aligned}\tilde{H}^*(X^{\mathbb{R}}_{E_6};\mathbb{C}) &\cong 36\tilde{H}^{*-1}(K_{\Lambda^1};\mathbb{C}) \oplus 27\tilde{H}^{*-1}(K_{\Lambda^1+\Lambda^4};\mathbb{C})\\ &\cong 36s^2\mathbb{C}^{49} \oplus 27(s\mathbb{C} \oplus s^2\mathbb{C}^{122})\end{aligned}$$

Concrete computation for type-A



Type-A case

We focus on the case

- $\Phi = \{e_i - e_j \mid 1 \leq i,j \leq n+1\} \subset \mathbb{R}^{n+1}$: the type A_n root system
- $\Delta = \{e_2 - e_1, e_3 - e_2, \ldots, e_{n+1} - e_n\}$
- $W = \mathcal{S}_{n+1}$
- K is the dual of the permutohedron

$$w\omega_i \to \{w(1), w(2), \ldots, w(i)\}$$

induces an $\mathcal{S}_{n+1}$-equivariant bijection between the vertices of K and non-empty proper subsets of $[n+1]$.
$J_1, J_2, \ldots, J_k \subset [n+1]$ form a simplex of K iff $J_1 \subset J_2 \subset \cdots \subset J_k$ up to a permutation

Type-A case

Fix a set of basis of $\mathbb{F}_2^n$

$$\epsilon_k := (1,k)\omega_1 \quad 1 \leq k \leq n,$$

where $(1,k) = s_{k-1}s_{k-2}\dots s_1 \in \mathcal{S}_{n+1}$ is the transposition.
Then, the columns of Λ are given by

$$[n+1] \supset J \leftrightarrow \begin{cases} \sum_{k\in J} \epsilon_k & (n+1 \notin J) \\ \sum_{k\notin J} \epsilon_k & (n+1 \in J) \end{cases}.$$

Note that $X_{A_n}^{\mathbb{R}} := M_{K,\Lambda}$ is the real locus of the (regular semisimple) Hessenberg variety, and also the real locus of the De Concini-Procesi wonderful model of the complement of the coordinate subspace arrangement in $\mathbb{C}^{n+1}$.

Type-A case

To compute

$$H^*(X^{\mathbb{R}}_{A_n};\mathbb{C}) \cong \bigoplus_{I \in Row(\Lambda)} \tilde{H}^{*-1}(K_I;\mathbb{C})$$

we have to know $Row(\Lambda)$.

We identify $Row(\Lambda)$ with even cardinality subsets of $[n+1]$:
For $Row(\Lambda) \ni \Lambda^{i_1} + \Lambda^{i_2} + \cdots + \Lambda^{i_k}$

- if k is even, associate $\{i_1, i_2, \dots, i_k\} \subset [n+1]$
- if k is odd, associate $\{i_1, i_2, \dots, i_k, n+1\} \subset [n+1]$

Now we investigate K_I for $I \in \binom{[n+1]}{2r}$.

Cohomology of type-A case

Theorem (Choi-Park)
For $I \in \binom{[n+1]}{2r}$,

$$K_I \simeq \bigvee^{a_{2r}} S^{r-1},$$

where a_{2r} is *Euler's zigzag number*.
As a corollary,

$$\beta_r(X^{\mathbb{R}}_{A_n}) = a_{2r}\binom{n+1}{2r}.$$

We would like to see the $\mathcal{S}_{n+1}$-module structure of $H^*(X^{\mathbb{R}}_{A_n};\mathbb{C})$.
For this, we construct a combinatorial model for $H^*(X^{\mathbb{R}}_{A_n};\mathbb{C})$.

Alternating sequence

For $I = \{p_1, p_2, \ldots, p_{2r}\} \in \binom{[n+1]}{2r}$, let

$$\mathrm{Seq}_I := \{\alpha\text{: permutation of } I\}$$

Denote its element by $(p_1 p_2 / \ldots / p_{2r-1} p_{2r})$. Set

$$\mathrm{Alt}_I := \{(p_1 p_2 / \ldots / p_{2r-1} p_{2r}) \mid p_1 < p_2 > p_3 < \cdots\}$$

Euler's zigzag number is by definition $a_{2r} = \#\mathrm{Alt}_I$ (for any I).

Cohomology of $X_{A_n}^{\mathbb{R}}$

Let $\mathbb{C}\langle \mathrm{Seq}_I \rangle$ be the free $\mathbb{C}$-module generated by Seq_I.
Define a submodule Q_I of $\mathbb{C}\langle \mathrm{Seq}_I \rangle$ generated by

1. $(p_1 \dots p_{2i-2}/p_{2i-1}p_{2i}/p_{2i+1} \dots p_{2r}) + (p_1 \dots p_{2i-2}/p_{2i}p_{2i-1}/p_{2i+1} \dots p_{2r})$
2. $(p_1 \dots /p_{2i-1}p_{2i}/p_{2i+1}p_{2i+2}/ \dots p_{2r}) + (p_1 \dots /p_{2i+1}p_{2i+2}/p_{2i-1}p_{2i}/ \dots p_{2r})$
   $+(p_1 \dots /p_{2i}p_{2i+1}/p_{2i-1}p_{2i+2}/ \dots p_{2r}) + (p_1 \dots /p_{2i-1}p_{2i+2}/p_{2i}p_{2i+1}/ \dots p_{2r})$
   $-(p_1 \dots /p_{2i-1}p_{2i+1}/p_{2i}p_{2i+2}/ \dots p_{2r}) - (p_1 \dots /p_{2i}p_{2i+2}/p_{2i-1}p_{2i+1}/ \dots p_{2r})$

   for $1 \le i \le r-1$

Theorem (Cho-Choi-K or S'[3], (c.f. Henderson))

$$H^r(X_{A_n}^{\mathbb{R}};\mathbb{C}) \cong \bigoplus_{I \in \binom{[n+1]}{2r}} \mathbb{C}\langle \mathrm{Seq}_I \rangle / Q_I$$ as $\mathcal{S}_{n+1}$-modules.

Furthermore, $\mathbb{C}\langle \mathrm{Seq}_I \rangle / Q_I \cong \mathbb{C}\langle \mathrm{Alt}_I \rangle$ as $\mathbb{C}$-modules.

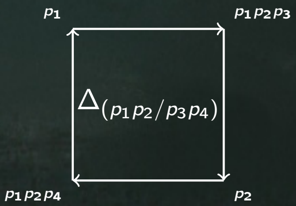
Sketch of proof

Define $\mathbb{C}\langle \mathrm{Seq}_I \rangle \to \tilde{H}^{r-1}(K_I; \mathbb{C})$ by sending $\alpha \in \mathrm{Seq}_I$ to the class dual to

$$\Delta_\alpha := \{p_1, p_2\} \star \{p_1 p_2 p_3, p_1 p_2 p_4\} \star \{p_1 p_2 p_3 p_4 p_5, p_1 p_2 p_3 p_4 p_6\} \cdots \subset K_I$$

where $\star$ is the join operation.

Example



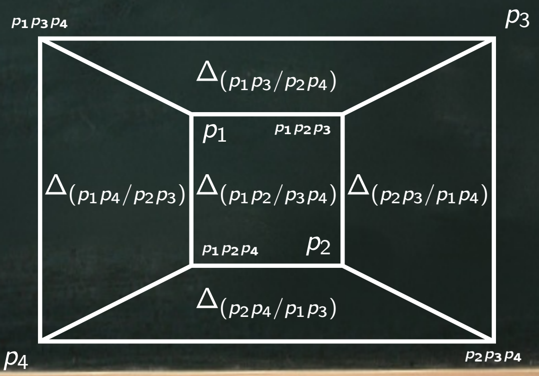
Sketch of proof

There are three steps:

1. $\phi : \bigoplus_{I \in \binom{[n+1]}{2r}} \mathbb{C}\langle \mathrm{Seq}_I \rangle \to H^r(X^{\mathbb{R}}_{A_n}; \mathbb{C})$ is surjective
2. $Q_I \subset \ker(\phi)$
3. $\dim(\mathbb{C}\langle \mathrm{Seq}_I \rangle / Q_I) \leq a_{2r}$

$Q_I \subset \ker(\phi)$

The first set of generators (relations) in Q_I comes from the reversing orientation.
The second set is explained by



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Future work

- Concrete computation of other types of root systems
- Ring structure of $H^*(X^{\mathbb{R}}_{A_n})$ in terms of sequences
- Computation for the orbifold $H^*(X^{\mathbb{R}}_{\phi}/W)$
- Examples of finite group representations on $H^*(M_{K,\Lambda})$

Thank you!