A product in equivariant homology for compact Lie group actions

Shizuo KAJI
Yamaguchi Univ. Japan / U. of Southampton UK

joint with Haggai Tene

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History

1. Tate cohomology: cup product in homology of finite groups
2. Kreck product: generalisation to homology of Lie groups
3. Our product: generalisation to equivariant homology of manifolds

Note: Greenlees-May[1995] defined a Tate cohomology spectra for compact Lie groups

Tate cohomology

- G: finite group
- $\to P_k \to P_{k-1} \to \cdots P_0 \to \mathbb{Z} \to 0$: the standard $\mathbb{Z}[G]$-resolution ($P_k := \mathbb{Z}[G^{k+1}]$)
- Then, $0 \to \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, \mathbb{Z}[G]) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, \mathbb{Z}[G]) \to \cdots$ is exact
- set $P_{-i} := \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i-1}, \mathbb{Z}[G])$ and

$$P_* := \to P_k \to P_{k-1} \to \cdots \to P_0 \to P_{-1} \to P_{-2} \cdots$$

- The *Tate cohomology* $\hat{H}^*(G) := H(\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, \mathbb{Z}))$

Tate cohomology

It is easy to see

$$\hat{H}^i(G) \simeq \begin{cases} H^i(G) & (i \geq 1) \\ \mathbb{Z}/|G|\mathbb{Z} & (i = 0) \\ 0 & (i = -1) \\ H_{-i-1}(G) & (i \leq -2). \end{cases}$$

It amalgamates homology and cohomology into one object.

Cup product in Tate cohomology

By $P_{i+j} \to P_i \otimes P_j$, the cup product is defined:

$$\hat{H}^l(G) \otimes \hat{H}^k(G) \to \hat{H}^{k+l}(G),$$

which gives a product in homology of degree $+1$:

$$H_{-l-1}(G) \otimes H_{-k-1}(G) \to H_{-k-l-1}(G)$$

through $\hat{H}^n(G) = H_{-n-1}(G)$ for $n \leq -2$. Note that we have to assume that $-l-1, -k-1 > 0$.

Stratifold homology

M. Kreck developed stratifold theory (a generalisation of manifold) to give a geometric definition of homology group as a bordism group. Based on this theory, he defined a product of degree $+\dim(G)+1$

$$H_k(BG;\mathbb{Z}) \otimes H_l(BG;\mathbb{Z}) \to H_{k+l+\dim(G)+1}(BG;\mathbb{Z})$$

for a compact Lie group G (with a mild condition).
Then, H. Tene proved

Theorem (Tene2012)

When G is finite, Kreck's product agrees with the cup product in the Tate cohomology.

Goal

The purpose of this talk is to

Extend the definition of the product to

$$H_k^G(M;R) \otimes H_l^G(M;R) \to H^G_{k+l+\dim(G)+1-\dim(M)}(M;R)$$

for G: a compact Lie group, M: a G-manifold, and R: a ring.

Our construction is a generalisation of that of Kreck's in that

- setting $M = pt$ recovers the product on $H_*(BG)$
- it doesn't use stratifold theory and purely homotopy theoretic
- it works with any coefficient ring R (while Kreck's definition only works for $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$)

Defining the product

The product

- R: the coefficient ring
- G: a compact Lie group and either connected or discrete (or more generally, the adjoint action of G on its Lie algebra is orientation preserving)
- M: a closed oriented manifold such that G acts on it smoothly and orientation preservingly

We will define:

$$H_k^G(M;R) \otimes H_l^G(M;R) \to H^G_{k+l+\dim(G)+1-\dim(M)}(M;R)$$
$$\alpha \otimes \beta \mapsto \alpha * \beta,$$

when $\dim(M) - \dim(G) < k, l$.

Remark on the degree

There are two justifications for the condition on the degree $\dim(M) - \dim(G) < k, l$.

- It is compatible with the classical setting of finite groups:

$$H_k(G) \otimes H_l(G) \to H_{k+l+1}(G), \quad (\dim(G) - \dim(pt) = 0 < k, l)$$

- When G is trivial, no product is defined.

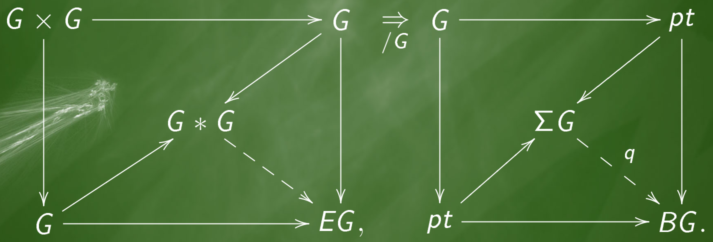
Remark

If we set $\mathbb{H}_i^G(M) = H^G_{i-\dim(G)+\dim(M)-1}(M)$, our product is of the form

$$\mathbb{H}_i^G(M) \otimes \mathbb{H}_j^G(M) \to \mathbb{H}_{i+j}^G(M) \qquad (i, j > 1)$$

Idea

The product is considered to be a generalisation of the Ganea construction: $G * G \to \Sigma G \to BG$



We regard $q_*[\Sigma G] \in H_{\dim(G)+1}(BG)$ as the product of the point class with itself.

Definition

Let $M_G := EG \times_G M$ be the Borel construction and
$H_*^G(M;R) := H_*(M_G;R)$ be the Borel equivariant homology.

Now, we will give the definition of

$$H_k^G(M;R) \otimes H_l^G(M;R) \to H^G_{k+l+\dim(G)+1-\dim(M)}(M;R)$$
$$\alpha \otimes \beta \mapsto \alpha * \beta,$$

when $\dim(M) - \dim(G) < k, l$.
Put $N = k + l + \dim(G) - \dim(M)$.

For $\alpha \in H_k^G(M; R)$, take

- S: k-skeleton of M_G
- $\iota_S : S \longrightarrow M_G$: a map
- $\hat{\alpha} \in H_k(S; R)$ such that $(\iota_S)_*(\hat{\alpha}) = \alpha$.

Take T similarly for $\beta \in H_l^G(M; R)$.

$$\begin{array}{ccccc} & & S & & \hat{\alpha} \\ & & \downarrow{\scriptstyle \iota_S} & & \big\downarrow \\ T & \xrightarrow{\iota_S} & M_G & & \alpha \end{array}$$

$$\hat{\beta} \longmapsto \beta$$

Consider the following pullback of fibrations:

$$
\begin{array}{ccccc}
G & = & G & = & G \\
\downarrow & & \downarrow & & \downarrow \\
Q & \longrightarrow & (M \times M)_G & \longrightarrow & BG \\
\downarrow \rho & & \downarrow & & \downarrow \Delta \\
S \times T & \xrightarrow{\iota_S \times \iota_T} & M_G \times M_G & \longrightarrow & BG \times BG
\end{array}
$$

Applying fibre integration to the left, we get

$$\rho^! : H_k^G(S) \otimes H_l^G(T) \to H^G_{k+l+\dim(G)}(Q)$$

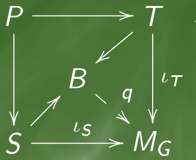
Remark on the fundamental class of a non-connected group G

Let $F \hookrightarrow E \to B$ be a fibration with $H_*(F;R) = H_{\leq n}(F;R)$. For $\pi_1(B)$-equvariant map $\chi : R \to H_n(F;R)$, the fibre integration is defined by

$$H_*(B;R) \xrightarrow{\chi} H_*(B;H_n(F;R)) \simeq E_2 \to E_\infty \simeq H_{*+n}(E;R).$$

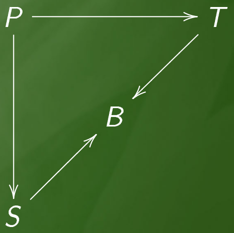
In our case, we need $R \to H_{\dim(G)}(G;R)$ which is invariant under the sandwich action of $\pi_1(BG \times BG) = \pi_0(G) \times \pi_0(G)$. There always exists the left $\pi_0(G)$-invariant fundamental class. If the conjugation action of $\pi_0(G)$ on $\mathrm{Lie}(G)$ is orientation preserving, the fundamental class becomes also right invariant.
Note: $O(2n)$ doesn't satisfy the condition.

Define P and B by the *pull-push construction*



,

where $\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow & & \downarrow \\ S & \longrightarrow & M_G \end{array}$ is homotopy pullback and



is homotopy pushout.

P and Q fit in the ladder of pullbacks

$$\begin{array}{ccc} P & \longrightarrow & M_G \\ \Big\downarrow \hat{\Delta}_G & & \Big\downarrow \Delta_G \\ Q & \longrightarrow & (M \times M)_G \\ \Big\downarrow \rho & & \Big\downarrow \\ S \times T & \xrightarrow{\iota_S \times \iota_T} & M_G \times M_G \end{array}$$

We have the Gysin map

$$\hat{\Delta}_G^! : H^G_{k+l+\dim(G)}(Q; R) \to H^G_{k+l+\dim(G)-\dim(M)}(P; R)$$

Finally, by the Mayer-Vietoris sequence

$$0 = H_{N+1}(S;R) \oplus H_{N+1}(T;R) \to H_{N+1}(B;R) \xrightarrow{\delta} H_N(P;R) \to H_N(S;R) \oplus H_N(T;R) = 0$$

(This is where we need $N = k + l + \dim(G) - \dim(M) > k, l$)

Composing everything to get

$$\begin{aligned} q_* \circ \delta^{-1} \circ \hat{\Delta}^!_G \circ \rho^! : & H^G_k(S) \otimes H^G_l(T) \to H^G_{k+l+\dim(G)}(Q) \\ & \to H^G_{k+l+\dim(G)-\dim(M)}(P) \\ & \to H^G_{k+l+\dim(G)-\dim(M)+1}(B) \\ & \to H^G_{k+l+\dim(G)-\dim(M)+1}(M_G) \end{aligned}$$

The images of $\hat{\alpha}, \hat{\beta}$ do not depend on the choices, and this gives a well-defined product.

Properties and Examples

First computation

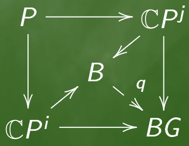
When $G = S^1, M = pt$

$$H_*^G(pt; R) = H_*(BS^1; R) \simeq R\langle a_{2k} \rangle \quad (k \geq 0),$$

where a_{2k} is represented by $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^\infty = BS^1$.

The product is given by

$$a_{2i} * a_{2j} = a_{2(i+j+1)}.$$



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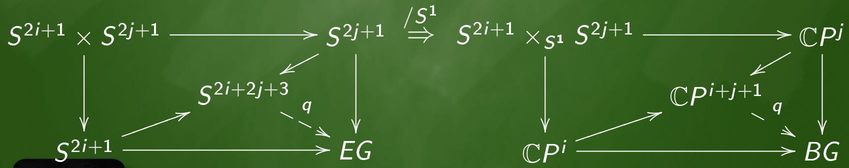

Functoriality

Proposition (Retriction)

For a closed subgroup inclusion $i : H \subset G$ and $\alpha, \beta \in H_*^G(M)$, we have

$$i^*(\alpha * \beta) = i^*(\alpha) * i^*(\beta) \in H_*^H(M)$$

Proposition (Inflation)

The product is compatible with the isomorphism:

$$H_*^H(M) \simeq H_*^G(M \times_H G)$$

Proposition (Change of coefficients)

The product is compatible with any morphism $R \to R'$ of the coefficients.

Product classes

Proposition (Vanishing for product classes)

Assume $G = G_1 \times G_2$ acting diagonally on $M = M_1 \times M_2$. The following composition vanishes

$$
\begin{aligned}
H_{k_1}((M_1)_{G_1}) \otimes H_{k_2}((M_2)_{G_2}) \otimes H_{l_1}((M_1)_{G_1}) \otimes H_{l_2}((M_2)_{G_2}) \\
\to H_{k_1+k_2}(M_G) \otimes H_{l_1+l_2}(M_G) \\
\to H_{k_1+k_2+k_1+k_2+\dim(G)+1}(M_G).
\end{aligned}
$$

for $k_i, l_i > \dim(M_i) - \dim(G_i)$

Product classes

Putting $M_1 = M_2 = pt$

Corollary

For $G = G_1 \times G_2$. The following composition vanishes:

$$\begin{aligned} H_{k_1}(BG_1) \otimes H_{k_2}(BG_2) \otimes H_{l_1}(BG_1) \otimes H_{l_2}(BG_2) & \\ \to H_{k_1+k_2}(BG) \otimes H_{l_1+l_2}(BG) & \\ \to H_{k_1+k_2+k_1+k_2+\dim(G)+1}(BG) & \end{aligned}$$

for $k_1, l_1, l_2 > -\dim(G)$.

Computation for classical groups

Proposition

The product vanishes for
$H_*(BU(n);\mathbb{Z})\ (n>2), H_*(BSp(n);\mathbb{Z})\ (n>1), H_*(BSO(n);\mathbb{Z})\ (n>3)$

Proof.
For $H_*(BU(n))\quad(n>2), H_*(BSp(n))\quad(n>1),$

- restriction $H_*(BG)\to H_*(BT^r)$ is injective
- the product vanishes on $H_*(BT^r)$ for $r>1$
- (for $H_*(BSO(n))$, also look at the restriction to the two torus)

□

Note: $BSU(2)=BSp(1)$ case is similar to BS^1.

Covering

Proposition (Covering)

Let $\phi : H \to G$ is an n-covering. For $\alpha, \beta \in H_*^H(M)$,

$$\phi_*(\alpha * \beta) = n\phi_*(\alpha) * \phi_*(\beta) \in H_*^G(M)$$

Corollary

For $H_*(BSO(3);\mathbb{Z}) \simeq \mathbb{Z}\langle b_{4k} \rangle \oplus$ 2-torsion $\quad (k \geq 0)$,

$$b_{4i} * b_{4j} = 2b_{4(i+j+1)}$$

and all the other products vanish.

Future work

- Compare to Greenlees-May's equivariant Tate cohomology
- Develop computational method
- Concrete computation for $H_*(BG)$
- Weaken conditions on G
- Extension to generalised homology theory
- Relation with other structures (co-product, Steenrod operations)
- Relation to string topology on BLG

Moltes grácies !