An Introduction to Schubert calculus

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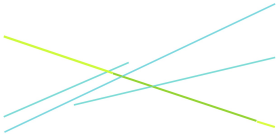
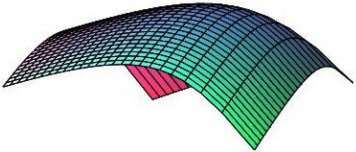
H. Schubert's quiz

(i) How many lines are there in 3-space
which meet all the four given lines ?



(ii) How many lines are there on a cubic-surface ?



H. Schubert asked in "Kalcul der abzählenden Géométrie" (1879)

Quiz in Combinatorics

Problem (Subset sum problem)

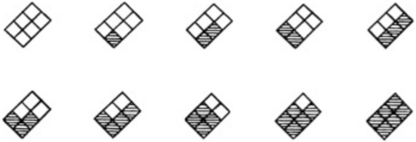
Find n distinct real numbers so that there are as many subsets as possible with the same sum.

Ex ($n = 6$): $1/2,\ \pi,\ \sqrt{2},\ 3/2,\ 2,\ 2-\pi-\sqrt{2}$
three subsets $\{2\}, \{1/2, 3/2\}, \{\pi, \sqrt{2}, 2-\pi-\sqrt{2}\}$ add up to 2.

Problem (Grid shading)

Fill an $n \times m$-rectangle titled by 45 degrees by k unit square boxes. Show that the number of ways is unimodal with respect to $0 \leq k \leq nm$.

Ex ($(n, m) = (2, 3)$):
$0 \leq 1 \leq 2 \leq 2 \geq 2 \geq 1 \geq 0$



They are all related and solved with Schubert calculus !

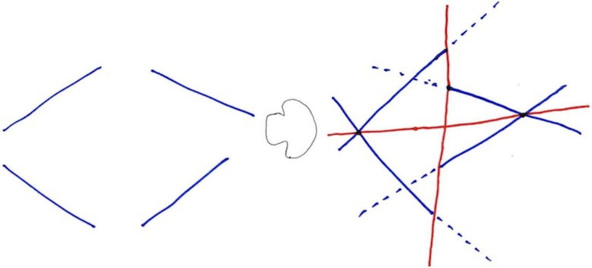
Answer for quiz (i) by H. Schubert

First, we'll see a naive argument by Schubert himself.
Assuming the following principle:

Conservation of number

The number is invariant if you change the configuration continuously under the condition that the number never becomes infinite

Schubert gave an answer for the quiz (i) as follows:



Schubert's "calculus"

Formalizing his argument,
he introduced the following symbols for conditions on a line:

$\Omega_{\emptyset}$: without any restriction

$\Omega_{(1)}$: intersecting a given line

$\Omega_{(2)}$: goes through a given point

$\Omega_{(1,1)}$: lying on a given plane

$\Omega_{(2,1)}$: lying on a given plane and goes through a given point on the plane

$\Omega_{(2,2)}$: lying on a given line, i.e. the line itself

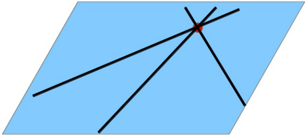


Figure: $\Omega_{(2,1)}$

(Here we don't need to specify the given point, line and plane, due to Conservation of Number.)

For example, the previous argument is described as $\Omega_{(1)} \cap \Omega_{(1)} = \Omega_{(1,1)} \cup \Omega_{(2)}$

Schubert's "calculus"

Then, the original quiz (i) translates to the following "calculation"

$$
\begin{aligned}
\Omega_{(1)} \cap \Omega_{(1)} \cap \Omega_{(1)} \cap \Omega_{(1)} &= (\Omega_{(1,1)} \cup \Omega_{(2)}) \cap (\Omega_{(1,1)} \cup \Omega_{(2)}) \\
&= (\Omega_{(1,1)} \cap \Omega_{(1,1)}) \cup 2(\Omega_{(1,1)} \cap \Omega_{(2)}) \cup (\Omega_{(2)} \cap \Omega_{(2)}) \\
&= \Omega_{(2,2)} \cup \emptyset \cup \Omega_{(2,2)} \\
&= \color{red}{2} \cdot \Omega_{(2,2)}
\end{aligned}
$$

How to make Schubert's argument rigorous ?

Hilbert's 15th problem

Problem

1. Give a rigorous foundation for those method
2. Give a systematic way (algorithm) for the calculation

1. What is *Conservation of number* ?
2. How to compute, for example,

$$\Omega_{\square} \cap \Omega_{\square} = \Omega_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \cup \Omega_{\square\square}$$

without relying on the geometric intuition ?

Ehreshmann's solution for part 1

Ehreshmann interpreted in the following setup:
Condition $\Leftrightarrow$ Subvariety of a moduli space
(Conjunction $\Leftrightarrow$ Intersection)

- $Gr_n(\mathbb{C}^{n+m})$: the moduli space of n-dimensional linear subspaces in $\mathbb{C}^{n+m}$, i.e., the *Grassmannian manifold*
- $\Omega_\lambda \subset Gr_n(\mathbb{C}^{n+m})$: Schubert variety indexed by
- $\lambda = (m \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0) \in \Lambda(n, m)$: (n, m)-partition (Young diagram)
- $\cap$: the cup (intersection) product in $H^*(Gr_n(\mathbb{C}^{n+m})) \quad (\simeq A^*(Gr_n(\mathbb{C}^{n+m})))$

In this setting, "Conservation of number" corresponds to the invariance of cup product.

Schubert variety

Definition

Let F_i be the i-dim subspace spanned by $e_1, \ldots, e_i$.

$$\Omega_\lambda^o = \{V \in Gr_n(\mathbb{C}^{n+m}) \mid \dim(V \cap F_{m+i-\lambda_i}) = i, \dim(V \cap F_{m+i-\lambda_i-1}) = i-1\} \cong \mathbb{C}^{|\lambda|}$$

$$\Omega_\lambda = \overline{\Omega_\lambda^o}$$

Theorem (Basis Theorem)

$$Gr_n(\mathbb{C}^{n+m}) = \bigcup_{\lambda \in \Lambda(n,m)} \Omega_\lambda^o$$

and

$$H^*(Gr_n(\mathbb{C}^{n+m})) = \mathbb{Z}\langle \Omega_\lambda \rangle$$

Schubert variety

A Schubert variety is represented by a matrix (spanned by row vectors).
Ex. Consider $Gr_2(\mathbb{C}^4)$: the space of lines in $\mathbb{C}P^3$ (by projectification)

$$\Omega^o_{\yng(1)} = \left\{ \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} \right\}, \Omega^o_{\yng(2)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix} \right\}$$

$$\Omega^o_{\yng(1,1)} = \left\{ \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix} \right\}, \Omega^o_{\yng(2,1)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \right\}$$

They correspond to sets of lines with some conditions

$\Omega_{\yng(1)}$ consists of "lines intersecting a given line"

$\Omega_{\yng(2)}$ consists of "lines going through a given point"

$\Omega_{\yng(1,1)}$ consists of "lines lying on a given plane"

Rephrasing the quiz (i) using Schubert variety

In $Gr_2(\mathbb{C}^4)$,

- $\Omega_{\yng(1)}$ consists of “lines intersecting a given line”
- $\Omega_{\yng(2,2)}$ consists of “lines included in a give line,” i.e., the line itself

Problem (rigorous one)

Determine the coefficient c in

$$\Omega_{\yng(1)} \cdot \Omega_{\yng(1)} \cdot \Omega_{\yng(1)} \cdot \Omega_{\yng(1)} = c \cdot \Omega_{\yng(2,2)} \in H^*(Gr_2(\mathbb{C}^4))$$

How to compute the intersection ?

Structure constant

The second part of Hilbert's problem is now rephrased as follows:

Problem

Give an effective algorithm for the structure constants

$$\Omega_\mu \cdot \Omega_\nu = \sum c_{\mu,\nu}^{\lambda} \Omega_\lambda$$

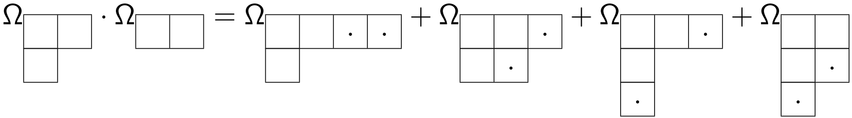
Structure constant

A partial answer is given by

Pieri formula

$$\Omega_\lambda \cdot \Omega_{\square\square\cdots\square\,k\text{-boxes}} = \sum \Omega_\mu,$$

where μ's are the diagrams obtained from λ by adding k-boxes at most one at a column.

Ex.


It is proven by inspecting the intersection carefully (linear algebra)

Pieri rule

Note that Pieri rule can be used for computing

$$\Omega_{(1,1)} \cdot \Omega_{(1,1)} = \Omega_{(1,1)} \cdot (\Omega_{(1)} \cdot \Omega_{(1)} - \Omega_{(2)})$$

$$= (\Omega_{(2,1)} + \Omega_{(1,1,1)}) \cdot \Omega_{(1)} - (\Omega_{(3,1)} + \Omega_{(2,1,1)})$$

$$= \Omega_{(2,1,1)} + \Omega_{(1,1,1,1)}$$

In fact, Pieri rule suffices to compute every products !

Schur polynomial

Young diagram $\Rightarrow$ symmetric polynomial

Definition (Schur polynomial(Jacobi-Trudi))

Let λ^t be the transposed Young diagram of $\lambda \in \Lambda(n, m)$.

$$s_\lambda = \det(c_{\lambda_i^t + j - i}),$$

where c_k is the k-th elementary symmetric polynomial in $\mathbb{Z}[t_1, \ldots, t_n]$.

Ex. $s_{\square\square} = \det \begin{pmatrix} c_1 & c_2 \\ c_0 & c_1 \end{pmatrix} = c_1^2 - c_2$

$s_{\boxed{k}\boxed{\cdots}} = h_k$ (the complete homogeneous polynomial)

$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \det \begin{pmatrix} c_2 & c_3 \\ 0 & c_0 \end{pmatrix} = c_2$, $s_{\begin{smallmatrix}\boxed{k}\\\boxed{\cdots}\end{smallmatrix}} = c_k$

Properties of Schur polynomial

- They have various equivalent definitions (e.g. Young tableaux)
- They are the characters of the irr-representations of $GL_n(\mathbb{C})$
- They form a basis for the invariant ring $I_n = \mathbb{Z}[t_1, \ldots, t_n]^{S_n}$
- the "1-row" family generates I_n as a ring
- They satisfy Pieri rule

$$s_\lambda s_{\square\square\cdots\square\, k\text{-boxes}} = \sum s_\mu,$$

Solution for part 2

Theorem

The following map is an injective homomorphism:

$$H^*(Gr_n(\mathbb{C}^{n+m})) \to I_n : \Omega_\lambda \mapsto s_\lambda$$

In particular, the structure constants can be calculated by *Littlewood-Richardson rule*.

Note that $R(S_n) \cong I_n : V_\lambda \mapsto s_\lambda$.

structure constants for Schubert varieties
=structure constants for Schur polynomials
=multiplicity of the tensor representation of the permutation group

Littlewood-Richardson rule

LR-rule descends from the *Tableaux ring*

Ex.

$$\begin{matrix} 1 & 1 & 2 \\ 3 & 4 & \\ 4 & & \end{matrix} \cdot \begin{matrix} 1 \\ 3 \end{matrix} = \begin{matrix} 1 & 1 & 2 & 3 \\ 3 & 4 & & \\ 4 & & & \end{matrix} \cdot \begin{matrix} 1 \end{matrix} = \begin{matrix} 1 & 1 & 1 & 3 \\ 3 & 4 & & \\ 4 & & & \end{matrix} \cdot_2 \begin{matrix} 2 \end{matrix}$$

$$= \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 4 & & \\ 4 & & & \end{matrix} \cdot_3 \begin{matrix} 3 \end{matrix} = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 4 & & \\ 3 & & & \end{matrix} \cdot_4 \begin{matrix} 4 \end{matrix}$$

$$= \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 4 & & \\ 3 & & & \\ 4 & & & \end{matrix}$$

How does Schur polynomial come into

Problem

Why Schur polynomial appears ?

The fiber bundle structure

$$Gr_n(\mathbb{C}^{n+m}) \hookrightarrow B(U(n) \times U(m)) \to BU(m+n)$$

gives a polynomial description of the cohomology ring

$$H^*(Gr_n(\mathbb{C}^{n+m})) \cong \frac{H^*(B(U(n) \times U(m)))}{(H^+(BU(m+n)))} \cong \frac{\mathbb{Z}[x_1, x_2, \dots, x_{n+m}]^{S_n \times S_m}}{(x_1, x_2, \dots, x_{n+m})^{S_{n+m}}}$$

(x_i's are the first Chern classes of the canonical lie bundle over BT^{n+m})

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(x_i's are the first Chern classes of the canonical lie bundle over BT^{n+m})

How does Schur polynomial come into

s_λ is a representative for Ω_λ in this presentation:

$$H^*(Gr_n(\mathbb{C}^{n+m})) \cong \frac{\mathbb{Z}[x_1, x_2, \ldots, x_{n+m}]^{S_n \times S_m}}{(x_1, x_2, \ldots, x_{n+m})^{S_{n+m}}}$$

$$\Omega_\lambda \leftrightarrow s_\lambda$$

Note: However, we don't know why Schur polynomial, the representation theoretic object, is related to the Schubert class. (One of the main open problems in Schubert Calculus)

Answer for quiz (ii) using Schur polynomials

Theorem: There 27 lines on a cubic-surface.

- Let γ be the dual to the canonical plane-bundle over $Gr_2(\mathbb{C}^4)$
- a section of $\mathrm{Sym}^k(\gamma)$ corresponds to a degree k function
- $$\begin{aligned}\text{lines on a cubic-plane} &= \text{zero locus of a generic section for } \mathrm{Sym}^3(\gamma)\\ &= \text{the top Chern class } c_4(\mathrm{Sym}^3(\gamma))\\ &= 3t_1(2t_1+t_2)(t_1+2t_2)3t_2\\ &= 9s_{(1,1)}(2s_{(1)}^2+s_{(1,1)})\\ &= {\color{red}27}\,s_{(2,2)}\end{aligned}$$
- Note: $c(\mathrm{Sym}^3(\gamma)) = c(L_1^3 \oplus L_1^2 \otimes L_2 \oplus L_1 \otimes L_2^2 \oplus L_2^3)$, where $c_1(L_i) = t_i$

Summary for $H^*(Gr_n(\mathbb{C}^{n+m}))$

The story for a Grassmaniann is summarized as

Topology	$\Leftrightarrow$	Combinatorics	$\Leftrightarrow$	Algebra
Schubert class	$\Leftrightarrow$	Young diagram	$\Leftrightarrow$	Schur polynomial

How can the theory be generalized ?

Schubert calculus for other Lie types

Recall that $Gr_n(\mathbb{C}^{n+m})$ is a projective homogeneous variety $\dfrac{U(n+m)}{U(n)\times U(m)}$

Replace

- $Gr_n(\mathbb{C}^{n+m}) \Rightarrow G/P$: projective homogeneous variety
- S_n: symmetric group $\Rightarrow W(G)$: Weyl group
- Young diagrams $\Rightarrow W(G)/W(P)$: coset of Weyl group (poset)
- Ω_λ: Schubert variety (indexed by Young diagram) $\Rightarrow \Omega_w = \overline{B^- wP}$: Schubert variety (indexed by the coset of Weyl group)
- s_λ: Schur polynomial $\Rightarrow \sigma_w$: Schubert polynomial

We consider the structure constant problem:

$$\Omega_u \cdot \Omega_v = \sum c_{u,v}^{w} \Omega_w$$

Schubert calculus for other Lie types

- G: a complex semi-simple simply connected algebraic group (e.g. SL_n)
- $B, (B^-)$: the (opposite) Borel subgroup (e.g. upper(lower)-triangular matrices)
- P: a parabolic subgroup, i.e., $G \supset P \supset B$ (e.g. $P = B$)
- $W(G), (W(P))$: the Weyl group of G (resp. P)
- G/P has a cell decomposition $G/P = \bigcup_{w \in W(G)/W(P)} B^- wP$
- the closure $\Omega_w = \overline{B^- wP}$ is a (possibly singular) subvariety called the Schubert variety
- $w \leq v$ by the *Bruhat order* on $W \Leftrightarrow \Omega_w \supset \Omega_v$
- $codim\, \Omega_w = 2l(w)$, two times the *length* of $w \in W$
- $\{\Omega_w \mid w \in W(G)/W(P)\}$ form a free basis for $H^*(G/P; \mathbb{Z})$ and are called the *Schubert classes*

$$H^*(G/P; \mathbb{Z}) = \bigoplus_{w \in W(G)/W(P)} \mathbb{Z}\langle \Omega_w \rangle$$

(odrinary) Schubert polynomial

By the fiber bundle $G/P \hookrightarrow BP \to BG$, we have

$$H^*(G/P;\mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, x_2, \ldots, x_r]^{W(P)}}{(\mathbb{Q}^+[x_1, x_2, \ldots, x_r]^{W(G)})},$$

A polynomial representative $\sigma_w(x)$ for Ω_w is called a *Schubert polynomial*.

- There is a choice up to the ideal.
- we have to take $\mathbb{Q}$-coefficients.
  However, this causes no problem when we consider the structure constant since $H^*(G/P;\mathbb{Z}) \subset H^*(G/P;\mathbb{Q})$.
- There are many known constructions:
  - Schur functions for $H^*(Gr(n, n+m))$
  - Lascoux-Schützenberger's Schubert polynomials for $H^*(GL_n(\mathbb{C})/B)$
  - Q- and $\tilde{Q}$-Schur functions for $H^*(Sp(n)/U(n))$ and $H^*(SO(2n+1)/U(n))$
  - BGG's polynomial for any Lie type

Equivariant Schubert calculus

The Schubert variety $\Omega_w = \overline{B^- wP} \subset G/P$ is T-invariant.
And hence, it also defines a class $\Omega_w^T \in H_T^{2l(w)}(G/P; \mathbb{Z})$.
Since $H_T^*(G/P; \mathbb{Z})$ is concentrated in even degrees, SSS collapses for

$$G/P \hookrightarrow ET \times_T G/P \to BT$$

and we have

$$H_T^*(G/P; \mathbb{Z}) = \bigoplus_{w \in W(G)/W(P)} H^*(BT; \mathbb{Z}) \langle \Omega_w \rangle$$

In this setting, we consider the structure constant problem:

$$\Omega_u^T \cdot \Omega_v^T = \sum c_{u,v}^w(t) \Omega_w^T, \quad c_{u,v}^w(t) \in H_T^*(pt) = H^*(BT)$$

Double Schubert polynomial

A polynomial representative $\mathcal{S}_w(t;x)$ for Ω_w^T in

$$\frac{\mathbb{Q}[t_1,t_2,\dots,t_r]\otimes\mathbb{Q}[x_1,x_2,\dots,x_r]^{W(P)}}{(f(t)\otimes 1-1\otimes f(x))}\cong H_T^*(G/P;\mathbb{Q}),$$

(f varies all the $W(G)$-invariants)

is called a *double Schubert polynomial*.

Note that:

- There is a choice up to the ideal.
- Putting $t_i=0$, we get an ordinary Schubert polynomial.
- we have to take $\mathbb{Q}$-coefficients.
  However, this causes no problem when we consider the structure constant since $H_T^*(G/P;\mathbb{Z})\subset H_T^*(G/P;\mathbb{Q})$.

Double Schubert polynomial

Using the classification of G, many people gave constructions, but only for the classical types.

- factorial Schur functions for $H_T^*(Gr(n, n+m))$
- Lascoux-Schützenberger's double Schubert polynomials for $H_T^*(GL_n(\mathbb{C})/B)$
- Kresch-Tamvakis polynomials for $H_T^*(G/B)$, where G is a classical group

However, there's no type-uniform construction in contrast to the ordinary case.

A relationship between double and ordinary

Theorem (K)

There is an explicit construction for double Schubert polynomials from ordinary ones.

This means:

- (Combining the Theorem by BGG) This gives a type-uniform construction for the double Schubert polynomials.
- The classes in the equivariant cohomology $\Omega_w^T \in H_T^*(G/P)$ can be determined by the ones in the ordinary cohomology $\Omega_v \in H^*(G/P)$ $(v \leq w)$.
- Applied to the known constructions for the ordinary Schubert polynomials, it produces double ones.
  (for example, Lascoux and Schützenberger's polynomials produces Lascoux and Schützenberger's double Schubert polynomials.)

A relationship between double and ordinary

The explicit construction is given as follows:
Let $P_i(w)$ for $w \in W$ to be

$$\{(w_1, w_2, \ldots, w_i) \in W^i \mid w_1 \cdot w_2 \cdots w_i = w, l(w_k) > 0, l(w_1) + \cdots + l(w_i) = l(w)\}$$

then a double Schubert polynomial $\mathcal{S}_w(t; x)$ is given as

$$\sum_{i=1}^{l(w)} \sum_{(w_1, w_2, \ldots, w_i) \in P_i(w)} (-1)^{i+1} \sigma_{w_1}(t) \sigma_{w_2}(t) \cdots \sigma_{w_{i-1}}(t)(\sigma_{w_i}(t) - \sigma_{w_i}(x)),$$

where σ_w is any representative for the ordinary Schubert class.

A relationship between double and ordinary

Sketch of proof:

- The torus fixed points G/P^T is a discrete set $\{wP/P \mid w \in W(G)/W(P)\}$.
- The inclusion map $G/P^T \hookrightarrow G/P$ induces an injection

$$H_T^*(G/P) \hookrightarrow H_T^*\left(G/P^T\right) \cong \bigoplus_{w \in W(G)/W(P)} H^*(BT)$$

- In terms of polynomial, $f(t;x) \mapsto \bigoplus f(t; w^{-1}t)$.
- The image of Ω_w^T is characterized combinatorially in terms of the Bruhat order on W.
  (GKM condition and the upper-triangularity)
- We can find a representative of Ω_w^T by a combinatorial argument.

Equivariant structure constant

Once we find Schubert polynomials, the problem of the equivariant structure constant becomes purely combinatorial.
However, this is still very difficult and there are few known results.

- Knutson and Tao's Puzzle rule for $H^*_T(Gr(n, n+m))$
- Morev's Littlewood?Richardson polynomials for $H^*_T(Gr(n, n+m))$
- Willems's "formula" for $H^*_T(G/B)$ for any Lie type

Corollary (K)

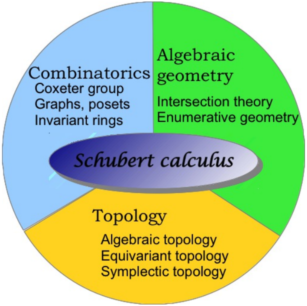
$$c^w_{u,v}(t) = (\Delta_w(\mathfrak{S}_u \mathfrak{S}_v))|_{x=t},$$

where Δ_w is the *divided difference operator* on x.
In particular, we have the equivariant Chevalley formula

$$\Omega^T_{s_i}\Omega^T_w = \sum_{\alpha\in\Phi, l(ws_\alpha)=l(w)+1} \frac{2\langle\alpha,\omega_i\rangle}{|\alpha|^2}\mathfrak{S}_{ws_\alpha} + (\omega_i(t) - \omega_i(w^{-1}t))\Omega^T_w$$

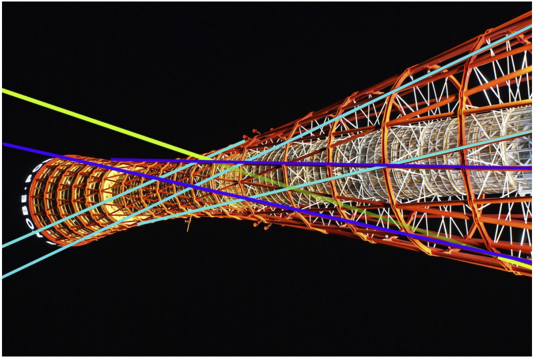
Directions of study

1. Structure constants for other Lie types (widely open except for $G = SL_n$)
2. Other cohomology theories including K-theory, MU-theory, quantum cohomology, and their torus equivariant versions
3. Extending Schubert calculus to wider classes of manifolds (e.g. symplectic, GKM)
4. Combinatorics of Coxeter group, tableaux, (co)invariant ring, and etc.
5. Application to Computer sciences (3D reconstruction, Network coding,...)



Explicit solution for 4-lines problem

For application to computer sciences, an explicit solution (rather than just the number of solutions) is often necessary.
Shabolovka tower can be used for the purpose.



- there is a hyperboloid which contains all the 3 given lines.
- the hyperboloid and the other line intersect at 2 points.
- the two lines on the hyperboloid going through the two intersection points give the answer.

(two purple lines give the answer)

Relation to combinatorics

Problem (Subset sum problem)

Find n distinct real numbers so that there are as many subsets as possible with the same sum.

Answer: $1, 2, \ldots, n$
(By the Spener property (Hard Lefschetz) for the cohomology of the isotropic Grassmaniann $Sp(n)/U(n)$)

Problem (Grid shading)

Fill an $n \times m$-rectangle titled by 45 degrees by k unit square boxes. The number of ways is unimodal with respect to $0 \leq k \leq nm$.

By the Spener property for the cohomology ring of the Grassmannian.

References

- Historical treatment
  Kleiman, S. L., *Problem 15: rigorous foundation of Schubert's enumerative calculus*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974), pp. 445–482.
- Combinatorial approach to Schubert calculus
  Fulton, W., *Young Tableaux: With Applications to Representation Theory and Geometry*, Cambridge University Press 1997.
  Hiller, H., *The geometry of Coxeter groups*, Research Notes in Mathematics, Vol. 54, Pitman Advanced Publishing Program 1982
- Geometry and topology of flag variety
  Kumar, S., *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, Progress in Mathematics vol. 204, Birkhauser
  Billey, S. and Lakshmibai, V., *Singular Loci of Schubert Varieties*, Progress in Mathematics 182, Birkhauser 2000

Thank you for listening

Большое спасибо