Representations on real toric manifolds

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Introduction

General goal

Realise interesting representations of finite groups G as (co)homology $H_*(M)$ of manifolds M

Our strategy

Combinatorics

Simplicial complex K
$\mathbb{F}_2$-characteristic function Λ

$\Leftrightarrow$

Topology

Real toric space $M_{K,\Lambda}$

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Simplicial complex K
$\mathbb{F}_2$-characteristic function Λ
Finite group $G \curvearrowright (K, \Lambda)$

$\Leftrightarrow$

Topology

Real toric space $M_{K,\Lambda}$
G-module $H_*(M_{K,\Lambda})$

Outline

- Real moment angle complex & Real toric space
- Combinatorial group actions on real toric spaces
- Ex 1. Real toric manifolds assoc. to root systems
- Ex 2. Real toric manifolds assoc. to nestohedra

Real moment angle complex
&
Real toric space

Real moment angle complex

K: simplicial complex over $[m] := \{1, 2, \ldots, m\}$

$$\mathbb{R}\mathcal{Z}_K = (\underline{D^1}, \underline{S^0})^K = \bigcup_{\sigma \in K} \left\{ (x_1, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma \right\}$$

is called the *real moment angle complex*.

$\mathbb{F}_2^m$ acts coordinate-wise on $\mathbb{R}\mathcal{Z}_K$.

Real toric space

$\Lambda\colon \mathbb{F}_2^m \to \mathbb{F}_2^n$: linear map (we often identify it with an $n \times m$-matrix)

$$M_{K,\Lambda} := \mathbb{R}\mathcal{Z}_K / \ker \Lambda \text{ is called the } \textit{real toric space}$$

Alternatively, we can start with any subgroup $H \subset \mathbb{F}_2^m$ and consider

$$M_{K,H} := \mathbb{R}\mathcal{Z}_K / H.$$

Example

- When Λ is the identity, $M_{K,\Lambda} = \mathbb{R}\mathcal{Z}_K$.
- When K is a polytopal $(n-1)$-sphere and the column vectors $\Lambda(i_1), \ldots, \Lambda(i_\ell)$ are linearly independent in $\mathbb{F}_2^n$ for any $\{i_1, \ldots, i_\ell\} \in K$, $M_{K,\Lambda}$ is nothing but the *small cover*.

Stable decomposition of Real moment angle complex

Let K_I denotes the restriction (full-subcomplex) of K on $I \subset [m]$.

Theorem (Bahri-Bendersky-Cohen-Gitler)

There exists a homotopy equivalence

$$\Sigma \mathbb{R}\mathcal{Z}_K \simeq \bigvee_{I \subset [m]} \Sigma^2 K_I$$

Stable decomposition of Real toric space

Theorem (Choi-K-Theriault or S^3)

Localised at an odd prime p or the rationals,

$$\Sigma(M_{K,\Lambda}) \simeq_p \bigvee_{I \in \mathrm{Row}(\Lambda)} \Sigma^2 K_I,$$

where $\mathrm{Row}(\Lambda) := \mathrm{Im}\Lambda^T \subset \mathbb{F}_2^m$ is naturally identified with a collection of subsets of $[m]$.

Example

CAUTION: $M_{K,\Lambda} = \mathbb{R}P^2$ for $K = \partial\Delta^2$ and $\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Each ΣK_I is contractible and the theorem does not hold for $p = 2$.

Group action on real toric space

Combinatorial group action on real toric space

- G: a finite group
- $G \curvearrowright K$: permutation of simplices

Then, G acts on $\mathbb{R}\mathcal{Z}_K$.

If in addition $\ker \Lambda$ is stable

$$\forall g \in G, \forall h = (h_1, \ldots, h_m) \in \ker \Lambda$$
$$gh = (h_{g(1)}, \ldots, h_{g(m)}) \in \ker \Lambda$$

then G acts on $M_{K,\Lambda}$.

The G-module $H_*(M_{K,\Lambda})$

In this case, the p-local homotopy equivalence

$$\Sigma(M_{K,\Lambda}) \simeq_p \bigvee_{I \in \mathrm{Row}(\Lambda)} \Sigma^2 K_I$$

turns out to induce a G-equivariant isomorphism on homology (by Ali Al-Raisi's thesis), so

Theorem

When $G \curvearrowright K$, the following are equivalent

1. $\ker \Lambda$ is G-stable
2. $\mathrm{Row}\Lambda$ is G-stable
3. the action induces one on $M_{K,\Lambda}$
4. $H_*(M_{K,\Lambda};\mathbb{Q}) \cong \bigoplus_{I \in \mathrm{Row}(\Lambda)} \tilde{H}_{*-1}(K_I;\mathbb{Q})$ as G-modules.

Example

- K: square on $\{1,2,3,4\}$
- $G = \langle g \mid g^4 = e \rangle$ acts on K by the cyclic permutation
$$g \cdot i = (i \mod 4) + 1$$
- $\Lambda_1 := \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$
- $\Lambda_2 := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

G does not preserve $\ker \Lambda_1$ but preserves $\ker \Lambda_2$.
The latter corresponds to $G \curvearrowright T^2 = M_{K,\Lambda_2}$ given by $g(x,y) = (-y,x)$.

Real toric manifold assoc. to root systems

Toric variety assoc to root system

For a root system $\Phi \subset \mathbb{R}^n$

- The Weyl group $W := \langle s_\alpha \mid \alpha \in \Phi \rangle$ acts on $\mathbb{R}^n$.
- The W-orbits of the fundamental co-weights define a polytope called the *Coxeter polytope*.
- Its boundary is a simplicial complex called the *Coxeter complex*. We denote it by K.
- The coordinates of the vertices define $\Lambda : V \to \mathbb{F}_2^n$.
- We denote $M_{K,\Lambda}$ by $X_\Phi^{\mathbb{R}}$. It is called the *real toric manifold associated to* Φ.

Remark

The complex toric manifolds associated to Φ have been studied by Procesi, Dolgachev-Lunts, Stembridge, Hiraku Abe, and others. We are interested in their real loci.

Type-A case

When Φ is of type A_n, the Coxeter complex K is the dual of the permutohedron of order $n+1$, acted by $W = \mathcal{S}_{n+1}$.

Remark

$X^{\mathbb{R}}_{A_n}$ is the real locus of the (regular semisimple) Hessenberg variety, and also the real locus of the De Concini-Procesi wonderful model of the complement of the coordinate subspace arrangement in $\mathbb{C}^{n+1}$.

To compute

$$H_*(X^{\mathbb{R}}_{A_n};\mathbb{Q}) \cong \bigoplus_{I \in \mathrm{Row}(\Lambda)} \tilde{H}_{*-1}(K_I;\mathbb{Q})$$

we have to know K_I for $I \in \mathrm{Row}(\Lambda)$.

- $\mathrm{Row}(\Lambda)$ is identified with $\{I \subset [n+1] \mid |I| : \text{even}\}$.
- The homotopy type of K_I is identified with that of a poset complex $\mathcal{B}_I^{odd}$ of the odd rank-selected Boolean poset on I.

The Boolean poset $\mathcal{B}_{2r}^{odd}$

$\mathcal{B}_I^{odd}$ is the poset whose elements are odd cardinality subsets of I, and the order is given by inclusion.
By Solomon (later by Stanley), its homology is computed:

$$\tilde{H}_*(\mathcal{B}_I^{odd}) = \begin{cases} S^{H_I} & (* = \left\lfloor \frac{|I|-1}{2} \right\rfloor) \\ 0 & \text{(otherwise)} \end{cases},$$

where S^{H_I} is the *skew Sphecht module* corresponding to the zigzag hook H of length $|I|$.

Homology of type-A varieties

Recall that the W-action has "two layers" in

$$H_*(X^{\mathbb{R}}_{A_n};\mathbb{Q}) \cong \bigoplus_{I\in \mathrm{Row}(\Lambda)} \tilde{H}_{*-1}(K_I;\mathbb{Q})$$

- "internal" action on K_I for $wI = I$
- "external" action permuting $I \to wI$.

Theorem (Cho-Choi-K or S'[3] (cf. Henderson))

$$H_r(X^{\mathbb{R}}_{A_n}) \simeq \mathrm{Ind}^{\mathcal{S}_{n+1}}_{\mathcal{S}_{\{1,\dots,2r\}}\times\mathcal{S}_{\{2r+1,\dots,n+1\}}}(S^{H_{[2r]}} \otimes S^{(n-2r)}),$$

where $S^{(n-2r)}$ is the trivial representation of $\mathcal{S}_{\{2r+1,\dots,n+1\}}$.

Homology of type-A varieties

Corollary

Let $Q = \{1, 3, \ldots, 2r-1\}$ and $c_{Q,\nu}$ be the number of standard tableaux of shape ν with descent set Q. Then, we have

$$H_r(X^{\mathbb{R}}_{A_n}) \cong \bigoplus_{\eta \vdash (n+1)} \left(\sum_{\nu} c_{Q,\nu} \right) S^{\eta},$$

where ν runs over all partitions of $2r$ that is contained in η, and η/ν has at most one box in each column.

Remark

As all irreducible modules are self-dual for Weyl groups, the above result also holds for cohomology.

Example: type-A_5

$H_3(X^{\mathbb{R}}_{A_5})$ is decomposed into irreducible $\mathcal{S}_6$ modules:

$$H_3(X^{\mathbb{R}}_{A_5}) \cong S^{(3,3)} \oplus 2S^{(3,2,1)} \oplus S^{(3,1,1,1)} \oplus S^{(2,2,2)} \oplus S^{(2,2,1,1)}$$

Standard tableaux of $\nu \vdash 6$ with descent set $\{1,3,5\}$ are

1	3	5
2	4	6

1	3	5
2	4	
6		

1	3	5
2	6	
4		

1	3	5
2		
4		
6		

1	3
2	5
4	6

1	3
2	5
4	
6	

$$\begin{aligned}\dim(\mathrm{S}^{(3,3)}) + 2\dim(\mathrm{S}^{(3,2,1)}) + \dim(\mathrm{S}^{(3,1,1,1)}) + \dim(\mathrm{S}^{(2,2,2)}) + \dim(\mathrm{S}^{(2,2,1,1)})\\ = 5 + 2 \times 16 + 10 + 5 + 9 = 61\end{aligned}$$

is the number of alternating permutations of length 6.

Homology of type-B varieties

For the type-B_r case, the Weyl group is the hyperoctahedral group W_{B_r}, whose irreducible modules are indexed by *double Young diagrams*.

Theorem (S'[3])

$$H_k(X^{\mathbb{R}}_{B_n}) \cong \bigoplus_{r\in\{2k-1,2k\}} \left(\mathrm{Ind}^{W_{B_n}}_{W_{B_r}\times W_{B_{n-r}}} \left(\bigoplus_{(\lambda,\mu)\vdash r} b(\lambda,\mu) S^{(\lambda,\mu)} \otimes S^{(\emptyset;(n-r))} \right)\right)$$

where $S^{(\emptyset;(n-r))}$ is the trivial representation of $W_{B_{n-r}}$.

Corollary

$$H_k(X^{\mathbb{R}}_{B_n}) \cong \bigoplus_{(\lambda,\nu)\vdash n} \left(\sum_{r\in\{2k-1,2k\}} \sum_{(\lambda,\mu)\vdash r,\mu\rightsquigarrow\nu} b(\lambda,\mu) \right) S^{(\lambda,\nu)}$$

(Co)homology of $X_\Phi^{\mathbb{R}}$

We can compute the integral cohomology of $X_\Phi^{\mathbb{R}}$ for almost all types; except for $\Phi = E_7$ and E_8.

Theorem ((S')^3 + (L. Cai-S. Choi) + (S. Choi-K-H. Park))

- $H^*(X_\Phi^{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) \cong H^{2*}(X_\Phi^{\mathbb{C}}; \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$ is well-known
- $H^*(X_\Phi^{\mathbb{R}}; \mathbb{Z})$ has no torsion other than 2-torsion
- $H^*(X_\Phi^{\mathbb{R}}; \mathbb{Q})$ is computed (computer assisted for exceptional types)

Example

We can utilise the symmetry: $K_I \simeq K_{wI}$ for $I \in \mathrm{Row}(\Lambda)$ and $w \in W$.

$$\begin{aligned}\tilde{H}^*(X_{E_6}^{\mathbb{R}}; \mathbb{C}) &\cong 27\tilde{H}^{*-1}(K_{\Lambda^1}; \mathbb{C}) \oplus 36\tilde{H}^{*-1}(K_{\Lambda^1+\Lambda^4}; \mathbb{C})\\ &\cong 36s\mathbb{C} \oplus 1323s^2\mathbb{C} \oplus 4392s^3\mathbb{C}\end{aligned}$$

Real toric manifold assoc. to nestohedra

Nestohedra

A *building set* $\mathfrak{B} \subset 2^{[n+1]} \setminus \{\emptyset\}$ on $[n+1]$ satisfies

1. $\{i\} \in \mathfrak{B}$ for any $i \in [n+1]$
2. if $I, J \in \mathfrak{B}$ and $I \cap J \neq \varnothing$, then $I \cup J \in \mathfrak{B}$.

If $[n+1] \in \mathfrak{B}$, then $\mathfrak{B}$ is said to be *connected*.

For a connected building set $\mathfrak{B}$, there is an associated polyhedron $P_{\mathfrak{B}}$ called the *nestohedron*.

Its boundary defines a simplicial complex $K_{\mathfrak{B}}$, and the coordinates of vertices define Λ.
Together, they give rise to a real toric manifold $X^{\mathbb{R}}_{\mathfrak{B}}$.

$\mathrm{Aut}(\mathfrak{B})$-module structure of $X_{\mathfrak{B}}^{\mathbb{R}}$

Lemma

Let $\mathfrak{B}$ be a connected building set. Then, $\mathrm{Aut}(\mathfrak{B})$ acts on $X_{\mathfrak{B}}^{\mathbb{R}}$ and we have an $\mathrm{Aut}(\mathfrak{B})$-module isomorphism

$$H_*(X_{\mathfrak{B}}^{\mathbb{R}}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda_{\mathfrak{B}})} \widetilde{H}_{*-1}\left((K_{\mathfrak{B}})_S\right).$$

In particular, let $Q_k = \{1, k+1, \dots, n+1\}$ and $\mathfrak{B}_{n,k} = \mathfrak{B}_{[n+1]}^{Q_k}$.
Then, $\mathrm{Aut}(\mathfrak{B}_{n,k}) = \mathcal{S}_{n+1}$.
This is a generalisation of $X_{A_n}^{\mathbb{R}}$ as $X_{\mathfrak{B}_{n,1}}^{\mathbb{R}} = X_{A_n}^{\mathbb{R}}$.

$\mathrm{Aut}(\mathfrak{B})$-module structure of $X_{\mathfrak{B}}^{\mathbb{R}}$

Theorem

For an odd integer n,

$$H_*(X^{\mathbb{R}}_{\mathfrak{B}_{n,k}}) \cong \begin{cases} 0 & (* > d) \\ S^{\lambda_Q} & (* = d), \end{cases}$$

where $d = \frac{n+1}{2} + \lfloor \frac{k}{2} \rfloor$, λ_Q is the skew hook with $n+1$ cells whose descent set is $\{1, 2, \ldots, k\} \cup \{1, 3, 5, \ldots, n\}$.

Skew Specht modules corresponding to skew hooks are called *Foulkes' module*. The above realises Foulkes' module of a special type.

Future work

- More examples of finite group representations on $H^*(M_{K,\Lambda})$
- Computation of $H^*(X_\Phi^{\mathbb{R}})$ for the remaining cases of $\Phi = E_7$ and E_8
- Ring structure of $H^*(X_\Phi^{\mathbb{R}})$
- Computation for the orbifold $H^*(X_\phi^{\mathbb{R}}/W)$

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Thank you!