Equivariant string products

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Outline of talk

- String products and Gysin maps
- An equivariant string product
- A secondary product

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String products and Gysin maps

String topology

In 1999 Chas and Sullivan discovered a rich algebraic structure on the homology of the free loop space LM over a closed oriented manifold M.

$$LM := \mathrm{Map}(S^1, M) := \{f : S^1 \to M\}$$

with the compact open topology.

Since then, algebraic structures on the (co)homology of free loop spaces have been actively studied under the name of *string topology*.

Free and based loop spaces

Let ΩM be the based loop space over M.
It admits a multiplication given by the concatenation of loops:

$$c : \Omega M \times \Omega M \to \Omega M$$

LM and ΩM are connected by the *evaluation fibration*:

$$\Omega M \to LM \xrightarrow{ev} M,$$

where $ev(f) = f(0)$ is the evaluation at the base point $0 \in S^1$.

String product (loop product)

Consider the following diagram on homology:

$$
\begin{array}{ccc}
H_s(\Omega M) \otimes H_t(\Omega M) & \xrightarrow{c_*} & H_{s+t}(\Omega M) \\
\downarrow & & \downarrow \\
H_s(LM) \otimes H_t(LM) & \overset{?}{\dashrightarrow} & H_{s+t-\mathrm{dim}(M)}(LM) \\
\downarrow {\scriptstyle ev_* \otimes ev_*} & & \downarrow {\scriptstyle ev_*} \\
H_s(M) \otimes H_t(M) & \xrightarrow{\Delta^!} & H_{s+t-\mathrm{dim}(M)}(M),
\end{array}
$$

where the multiplication on the fibre is induced by concatenation, and on the base by the *intersection product* $\Delta^!$.
Chas-Sullivan's string product fills the dotted arrow on the total spaces.

Chas-Sullivan's String product

Consider the following diagram with a pullback square:

$$\begin{array}{ccccc} LM & \xleftarrow{\;c\;} & \mathrm{Map}(S^1 \vee S^1, M) & \xrightarrow{\;\hat{\Delta}_M\;} & LM \times LM \\ & & \downarrow & & \downarrow{\scriptstyle ev \times ev} \\ & & M & \xrightarrow{\;\Delta_M\;} & M \times M \end{array}$$

where c is the concatenation of "8-shaped" loops. Chas-Sullivan's string product $H_s(LM) \otimes H_t(LM) \to H_{s+t-\dim(M)}(LM)$ is the composition of the *Gysin map* $\hat{\Delta}^!_M$ and c_*.
(actually, this is Cohen-Jones's definition)

String product in BG

Let G be a compact Lie group and BG be its classifying space.

(when G is non-connected, we assume $Ad(G)$ acts on the Lie algebra orientation preservingly.)

We can replace M with BG in the previous diagram:

$$\begin{array}{ccccc} LBG & \xleftarrow{c} & \mathrm{Map}(S^1 \vee S^1, BG) & \xrightarrow{\hat{\Delta}_G} & LBG \times LBG \\ & & \downarrow & & \downarrow{\scriptstyle ev \times ev} \\ & & BG & \xrightarrow{\Delta_G} & BG \times BG \end{array}$$

Chataur-Menichi's string product

$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)}(LBG)$ is the composition of the Gysin map $\hat{\Delta}_G^!$ with the concatenation of loops c.

Gysin maps

In the definitions, Gysin maps (or pushforward/umkehr/wrong-way maps) play the key role. Two different kinds are used:

$$H_*(LM \times LM) \xrightarrow{\hat{\Delta}^!_M} H_{*-\dim(M)}\mathrm{Map}(S^1 \vee S^1, M)$$

(the *intersection product* or the Gysin map associated to a finite codimensional embedding)

$$H_*(LBG \times LBG) \xrightarrow{\hat{\Delta}^!_G} H_{*+\dim(G)}\mathrm{Map}(S^1 \vee S^1, BG)$$

(the *Grothendieck bundler transfer* associated to a fibre bundle with a finite dimensional fibre having a nice Lie group action)

Equivariant String product

Equivariant string product

We would like to combine the two string products.
For this, we work in an equivariant setting.
Let G act on M orientation preservingly and $M_G := EG \times_G M$ be the Borel construction. We will define

$$H_s(L(M_G)) \otimes H_t(L(M_G)) \to H_{s+t+\dim(G)-\dim(M)}(L(M_G)),$$

which reduces

- to Chas-Sullivan's when $G = *$, and
- to Chataur-Menichi's when $M = *$.

Idea of Definition

As before, consider the following pullback

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, M_G) & \xrightarrow{\hat{\Delta}_{M_G}} & L(M_G) \times L(M_G) \\ \downarrow & & \downarrow{\scriptstyle ev \times ev} \\ M_G & \xrightarrow{\Delta_{M_G}} & M_G \times M_G \end{array}$$

However, the problem now is that we do not have a Gysin map for $\hat{\Delta}_{M_G}$ since its fibre is not finite dimensional and it is not a finite codimensional embedding.

We will circumvent this difficulty by decomposing the diagonal map into two steps and use the Gysin maps step-by-step.

Idea of Definition

The diagonal map $\Delta_{M_G} : M_G \to M_G \times M_G$ decomposes into two steps:

$$M_G \xrightarrow{\Delta_G} (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G,$$

where Δ_G is the equivariant diagonal (which is codimension $\dim(M)$) and

$$(G \times G)/\Delta G \hookrightarrow (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G$$

is the homogeneous fibration with fibre G. That is, the pullback of

$$(G \times G)/\Delta G \hookrightarrow E(G \times G)/\Delta G \to E(G \times G)/(G \times G)$$

Thus, both Δ_G and q have Gysin maps.

Idea of Definition

Consider the ladder of pullbacks:

$$\begin{array}{ccccc}
L(M_G) \times_{M_G} L(M_G) & \xrightarrow{\hat{\Delta}_G} & L(M_G) \times_{BG} L(M_G) & \xrightarrow{\hat{q}} & L(M_G) \times L(M_G) \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle ev \times ev} \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G & \xrightarrow{q} & M_G \times M_G .
\end{array}$$

Note that $\mathrm{Map}(S^1 \vee S^1, M_G) \simeq L(M_G) \times_{M_G} L(M_G)$. Then define the equivariant string product as the composition:

$$\begin{aligned}
H_s(L(M_G)) \otimes H_t(L(M_G)) &\to H_{s+t}(L(M_G) \times L(M_G)) \\
&\xrightarrow{\hat{q}^!} H_{s+t+\dim(G)}(L(M_G) \times_{BG} L(M_G)) \\
&\xrightarrow{\hat{\Delta}^!_{\boldsymbol{G}}} H_{s+t+\dim(G)-\dim(M)}(L(M_G) \times_{M_{\boldsymbol{G}}} L(M_G)) \\
&\xrightarrow{c_*} H_{s+t+\dim(G)-\dim(M)}(L(M_G))
\end{aligned}$$

Gysin map

For the definition, we constructed suitable Gysin maps.

Proposition

There are Gysin maps for the lifts $\hat{q}$ and $\hat{\Delta}_G$ in the previous slide. They are compatible with the group restriction and the group induction w.r.t. closed subgroups $H \subset G$.

String coproduct in M_G

Similarly, we can define an equivariant string co-product. Consider

$$\begin{array}{ccccc} L(M_G)\times L(M_G) & \xleftarrow{i} & \mathrm{Map}(S^1\vee S^1, M_G) & \xrightarrow{\hat{\Delta}_{M_G}} & L(M_G) \\ & & \downarrow & & \downarrow{\scriptstyle (ev_0, ev_{1/2})} \\ & & M_G & \xrightarrow{\Delta_{M_G}} & M_G\times M_G, \end{array}$$

where i is the inclusion. We obtain

$$H_*(L(M_G)) \xrightarrow{\hat{\Delta}^!_{M_G}} H_{*+\dim(G)-\dim(M)}(\mathrm{Map}(S^1\vee S^1, M_G)) \xrightarrow{i_*} H_{*+\dim(G)-\dim(M)}(L(M_G)\times L(M_G)).$$

Remark

There are other approaches for equivariant string topology:

- Behrend-Ginot-Noohi-Xu constructed a similar product in the language of stacks.
- Félix-Thomas constructed one for Gorenstein spaces on the rational homology
- Lupercio-Uribe-Xicotencatl constructed one on the rational homology for orbifolds $[M/G]$ when G is finite.
- Ángel-Backelin-Uribe extended above to the integer coefficients.

We do not know whether our string product coincides with theirs or not.

A secondary product

Vanishing of Chataur-Menichi's

In our framework, it is easily shown that

Proposition

Chataur-Menichi's product on $H_*(LBG)$ vanishes in positive degrees.

In fact, we can show

Theorem

Let $F_{g,p+q}$ be the surface of genus g with *p-incoming* and *q-outgoing* boundary circles. Chataur-Menichi's *string operations* assoc. to $F_{g,p+q}$

$$\mu(F_{g,p+q}) : H_*(LBG)^{\otimes p} \to H_{\dim(G)(2g+p+q-2)}(LBG)^{\otimes q}$$

is trivial unless $g = 0$ and $p = 1$, or $* = 0$.

$\mu(F_{\mathbf{g},\mathbf{p}+\mathbf{q}})$ reduces to the product when $g = 0, p = 2, q = 1$, and to the co-product when $g = 0, p = 1, q = 2$

Secondary product

Vanishing of

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)}(LBG).$$

for $s, t > 0$ suggests that we may define a "secondary" product. In fact, we can define for $s, t > 0$

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)+1}(LBG),$$

This product does not usually vanish!

Theorem

When the negative Tate cohomology of G has non-vanishing cup product, the secondary string product for LBG is not trivial.

Secondary product

We omit how to define it, which is technically involved. Our framework is in a fairly general setting. Given a homotopy pullback square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & M_G \end{array}$$

we defined

$$H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)+1}(P).$$

for $s > F_X - \dim(G)$ and $t > F_Y - \dim(G)$, where F_X is the homological dimension of the homotopy fibre of $X \to M_G$.

Application: Tate cohomology

The identity (pullback) square

$$\begin{array}{ccc} M_G & = & M_G \\ \| & & \| \\ M_G & = & M_G, \end{array}$$

gives rise to a *secondary equivariant intersection product*

$$H_s^G(M) \otimes H_t^G(M) \to H^G_{s+t+\dim(G)-\dim(M)+1}(M).$$

Theorem

When $M = pt$ and G is finite, this coincides with the cup product in negative Tate cohomology of G.

Future work

- Develop computational method/algebraic treatment
  - via Eilenberg-Moore spectral sequence
  - via rational homotopy theory
- Define other stringy operations (e.g. BV-structure)
- Extend the definition to more general groups (e.g., p-compact groups)

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Merci

Thank you!

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