External products in equivariant homology

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Outline

The goal of this talk is to generalise the external product

$$\times : H_s(X) \otimes H_t(Y) \to H_{s+t}(X \times Y).$$

1. Two external products in the homology of a fibre square
2. An equivariant external product in the homology of a fibre square
3. A secondary product and its applications
4. Computational example

External product for fibre square

Given a pullback of fibre bundles (or a homotopy pullback)

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

with certain conditions on B.
We will define homomorphisms of the form

$$H_s(X) \otimes H_t(Y) \to H_{s+t+*}(X \times_B Y)$$

with degree shifts, which reduce to the ordinary external product when $B = pt$.

External product for fibre square

The key idea is that a homotopy pullback

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

is equivalent to the pullback of fibrations

$$\begin{array}{ccccc} \Omega B & \longrightarrow & X \times_B Y & \xrightarrow{\hat{\Delta}} & X \times Y \\ \| & & \downarrow & & \downarrow f \times g \\ \Omega B & \longrightarrow & B & \xrightarrow{\Delta} & B \times B \end{array}$$

and we use various *wrong-way maps* $\Delta^* : H_*(B \times B) \to H_{*+\mathrm{shift}}(B)$.

Generalised external product of first type

External product for fibre square

We define two kinds of external products.
First, let $B = BG$ be the classifying space of a compact Lie group G.
We require a technical condition: The conjugation action of G on $H_{\dim(G)}(G)$ preserves the fundamental class. This is automatically satisfied when G is connected or finite.
We have the (dual) *fibre integration*:

$$\begin{aligned}\hat{\Delta}^{\natural} : H_{s+t}(X \times Y) \to H_{s+t}(X \times Y; H_{\dim(G)}(G)) \simeq E^2_{s+t,\dim(G)} \\ \to E^{\infty}_{s+t,\dim(G)} \simeq H_{s+t+\dim(G)}(X \times_{BG} Y).\end{aligned}$$

Composing with the ordinary external product, we define

$$\varphi_{BG} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)}(X \times_{BG} Y)$$

Application: String product in BG

Consider the following pullback diagram

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, BG) & \longrightarrow & LBG \\ \big\downarrow & & \big\downarrow {\scriptstyle ev} \\ LBG & \xrightarrow{\;ev\;} & BG \end{array}$$

where $LBG \simeq \mathrm{Map}(S^1, BG) \simeq BLG$.
Composing the first external product with the concatenation of loops $\mathrm{Map}(S^1 \vee S^1, BG) \to LBG$, we obtain

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)}(LBG)$$

which is equivalent to the Chataur-Menichi product.

Generalised external product of second type

External product for fibre square

Second, let $B = M$ be an oriented closed manifold. We have the *Gysin map* covering the diagonal $\Delta : M \to M \times M$:

$$\hat{\Delta}^! : H_{s+t}(X \times Y) \to H_{s+t-\dim(M)}(X \times_M Y).$$

Composing with the ordinary external product, we define

$$\varphi_M : H_s(X) \otimes H_t(Y) \to H_{s+t-\dim(M)}(X \times_M Y).$$

Note that when $X = Y = M$, it restricts to the intersection product

$$H_s(M) \otimes H_t(M) \to H_{s+t-\dim(M)}(M).$$

Application: String product in M

Consider the following pullback

$$\begin{array}{ccc} \mathrm{Map}(S^1 \vee S^1, BG) & \longrightarrow & LM \\ \downarrow & & \downarrow{\scriptstyle ev} \\ LM & \xrightarrow{ev} & M \end{array}$$

Composing the second external product with the concatenation of loops, we obtain

$$H_s(LM) \otimes H_t(LM) \to H_{s+t-\dim(M)}(LM),$$

which is equivalent to the Chas-Sullivan product.

Two external products

When B is the classifying space of a compact Lie group

$$\varphi_{BG} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)}(X \times_{BG} Y)$$

When M is a closed oriented manifold

$$\varphi_M : H_s(X) \otimes H_t(Y) \to H_{s+t-\dim(M)}(X \times_M Y)$$

Can we unify the two constructions?

Equivariant generalised external product

Equivariant external product

We combine the previous two constructions and define an external product in an equivariant setting. Let G acts on M orientation preservingly and M_G be the Borel construction. For a homotopy pullback

$$\begin{array}{ccc} X \times_{M_G} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M_G \end{array}$$

we will define

$$\varphi_{M_G} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

Definition: Equivariant product

The problem is, we cannot define a wrong-way map for the diagonal map

$$M_G \to M_G \times M_G$$

since it does not have finite dimensional homotopy fibre nor finite codimension.

We will decompose the diagonal map into two steps and define wrong-way maps step-by-step.

Definition: Equivariant product

The diagonal map $M_G \to M_G \times M_G$ decomposes into two steps:

$$M_G \xrightarrow{\Delta_G} (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G,$$

where Δ_G is the equivariant diagonal (which is codimension $\dim(M)$) and

$$(G \times G)/\Delta G \hookrightarrow (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G$$

is the homogeneous fibration. That is, the pullback of

$$(G \times G)/\Delta G \hookrightarrow E(G \times G)/\Delta G \to E(G \times G)/(G \times G)$$

Definition: Equivariant product

Define Q as in the ladder of pullbacks:

$$
\begin{array}{ccccc}
X \times_{M_G} Y & \xrightarrow{\hat{\Delta}_G} & Q & \xrightarrow{\hat{q}} & X \times Y \\
\downarrow & & \downarrow & & \downarrow \\
M_G & \xrightarrow{\Delta_G} & (M \times M)_G & \xrightarrow{q} & M_G \times M_G .
\end{array}
$$

Then define the product φ_{M_G} as the composition:

$$
\begin{aligned}
H_s(X) \otimes H_t(Y) &\to H_{s+t}(X \times Y) \\
&\xrightarrow{\hat{q}^{\natural}} H_{s+t+\dim(G)}(Q) \xrightarrow{\hat{\Delta}_G^{!}} H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).
\end{aligned}
$$

Equivariant product unifies the two

The equivariant external product

$$\varphi_{M_G} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y)$$

restricts to

$$\varphi_{BG} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)}(X \times_{BG} Y)$$

when $M = pt$ and

$$\varphi_M : H_s(X) \otimes H_t(Y) \to H_{s+t-\dim(M)}(X \times_M Y).$$

when $G = *$.

Vanishing of φ_{M_G}

An easy but interesting property of φ_{M_G} is that it vanishes in higher degrees:

Theorem

Let F_X (resp. F_Y) be the homological dimension of the homotopy fibre of the composition $X \to M_G \to BG$. Then,

$$\varphi_{M_G} : H_s(X; R) \otimes H_t(Y; R) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y; R)$$

vanishes if $s > F_X - \dim(G)$ or $t > F_Y - \dim(G)$.

When applied to special cases, it has non-trivial consequences.

Application of the external product

Chataur-Menichi defined *string operations* (or HCFT) for LBG:
For a surface $F_{g,p+q}$ of genus g with *p-incoming* and *q-outgoing* boundary circles, they defined a homomorphism

$$\mu(F_{g,p+q}) : H_*(LBG)^{\otimes p} \to H_{\dim(G)(2g+p+q-2)}(LBG)^{\otimes q}$$

which is compatible with the gluing of the surfaces.
(when $g = 0, p = 2, q = 1$, it gives the product we saw earlier)
A consequence of our vanishing theorem is:

Corollary

$\mu(F_{g,p+q})$ is trivial unless $g = 0$ and $p = 1$, or $* = 0$.

Secondary product

Secondary product

Vanishing of

$$\varphi_{M_G} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

for $s > F_X - \dim(G)$ or $t > F_Y - \dim(G)$ suggests that we may define a "secondary" product.
In fact, we can define

$$\psi : H_s(X) \otimes H_t(Y) \to H^G_{s+t+\dim(G)-\dim(M)+1}(M).$$

for $s > F_X - \dim(G)$ and $t > F_Y - \dim(G)$.

Secondary product

In the definition of the equivariant external product, we had the following pullback:

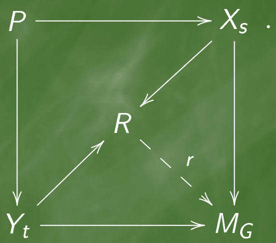
$$\begin{array}{ccc} X \times_{M_G} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M_G\,. \end{array}$$

Pulling back further to the skeletons X_s and Y_t, we define

$$\begin{array}{ccc} P & \longrightarrow & X_s \\ \downarrow & & \downarrow \\ Y_t & \longrightarrow & M_G\,. \end{array}$$

Secondary product

Now we take the homotopy pushforward of the upper-left corner



Our secondary product is defined to be the composition

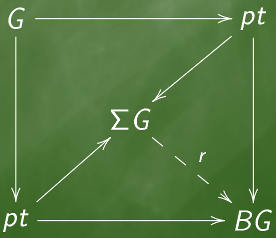
$$\psi : H_s(X_s) \otimes H_t(Y_t) \xrightarrow{\varphi_{M_G}} H_{s+t+\dim(G)-\dim(M)}(P)$$
$$\xrightarrow{\text{sus}} H_{s+t+\dim(G)-\dim(M)+1}(R) \xrightarrow{r_*} H_{s+t+\dim(G)-\dim(M)+1}(M_G).$$

Simplest example

Let $X = Y = M = pt$. Then the defining pull-push diagram for

$$\psi : H_0(pt) \otimes H_0(pt) \to H_{\dim(G)+1}(BG)$$

is identified with the first stage of the Ganea construction:



Application: String Product in BG

When $X = Y = M_G$, it specialises to a product in $H_*^G(M)$:

$$\psi : H_s^G(M) \otimes H_t^G(M) \to H^G_{s+t+\dim(G)-\dim(M)+1}(M).$$

Let G be a compact Lie group acting on itself by conjugation. It is well-known that the Borel construction $G_G^{ad} \simeq LBG \simeq BLG$.
In this case, ψ gives rise to

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)+1}(LBG),$$

which is a secondary product of Chataur-Menichi's string product.
This product does not usually vanish!

Application: Tate cohomology

When $M = pt$, we have

$$H_s(BG) \otimes H_t(BG) \to H_{s+t+\dim(G)+1}(BG)$$

Theorem

It coincides with the product in Tate cohomology when G is finite.

That is, ψ generalises Tate cohomology ring in two ways:

- G can now be not only a finite group but a compact Lie group
- the group homology (the equivariant homology of a point) is replaced by the equivariant homology of a manifold

Computational Examples

First computation

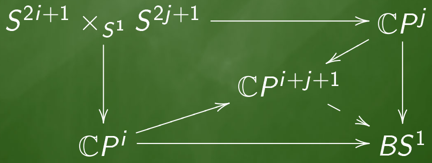
When $G = S^1, M = pt$

$$H_*^G(pt;\mathbb{Z}) = H_*(BS^1;\mathbb{Z}) \simeq \mathbb{Z}\langle a_{2k}\rangle \quad (k \geq 0),$$

where a_{2k} is represented by $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^\infty = BS^1$.

The product $H_s(BS^1) \otimes H_t(BS^1) \to H_{s+t+2}(BS^1)$ is given by

$$a_{2i} * a_{2j} = a_{2(i+j+1)}.$$



Computation for classical groups

Proposition

The product vanishes for all compact connected classical Lie groups of rank greater than 1.
For $H_*(BSp(1);\mathbb{Z}) \simeq \mathbb{Z}\langle a_{4k}\rangle \quad (k \geq 0)$,

$$a_{4i} * a_{4j} = a_{4(i+j+1)}.$$

For $H_*(BSO(3);\mathbb{Z}) \simeq \mathbb{Z}\langle b_{4k}\rangle \oplus \text{2-torsion} \quad (k \geq 0)$,

$$b_{4i} * b_{4j} = 2b_{4(i+j+1)}$$

and all the other products vanish.

Computation for $\mathbb{C}P^1$

Let $S^1 \curvearrowright \mathbb{C}P^1$ by the standard action. Then,
$H_*^{S^1}(\mathbb{C}P^1) = \mathbb{Z}\langle \alpha_{2k}, \beta_{2k+2} \rangle$, where

$$\alpha_{2k} : S^{2n+1} \times_{S^1} pt \to ES^1 \times_{S^1} \mathbb{C}P^1$$
$$\beta_{2k+2} : S^{2n+1} \times_{S^1} \mathbb{C}P^1 \to ES^1 \times_{S^1} \mathbb{C}P^1$$

$\psi : H_s^{S^1}(\mathbb{C}P^1) \otimes H_t^{S^1}(\mathbb{C}P^1) \to H_{s+t}^{S^1}(\mathbb{C}P^1)$ is computed as

Proposition

$\alpha_{2i} * \alpha_{2j} = 0, \alpha_{2i} * \beta_{2j+2} = \alpha_{2(i+j+1)}, \beta_{2i+2} * \beta_{2j+2} = \beta_{2(i+j+1)+2}$
We can show this by *localisation to the fixed points* arguments.

Future work

- Find other applications
- Develop computational method
  - secondary product for $H_*(BG)$
  - secondary product for $H_*^T(M)$ for toric, flag, and GKM manifolds
- Extension to generalised homology theory
- Relation with other structures (co-product, Steenrod co-operations)
- Compare the secondary product to Greenlees-May's Tate cohomology

Thank you very much!

Большое спасибо