# Lie group action on a GKM manifold

Shizuo KAJI
U. of Southampton / Yamaguchi Univ.

Topology Seminar
Southampton, Dec. 9, 2013

No Michael Jackson
nor Penguins today.
Sorry !

# Motivation

*Equivariant topology* is the study of spaces with group symmetry. For a group $G$, we investigate the category $\mathcal{T}op_G$ consisting of

- spaces with $G$-action (called *G-spaces*): $\exists G \times M \to M$
- maps between $G$-spaces which commutes with the actions (called *G-equvariant maps*) $f : M \to N, \quad f(gm) = gf(m)$

through functors such as

$$H^*_G(-) : \mathcal{T}op_G \to \mathcal{CGA}.$$

The group symmetry often simplifies things drastically.

# Motivation

Especially, when the group is abelian (i.e., a torus)

- There are interesting classes of manifolds with "nice" torus actions such as
  - *flag manifold*: the homogeneous space $G/P$ of a simple Lie group $G$ divided by its parabolic subgroup $P$. (ex. complex Grassmanian )
  - (smooth) *toric manifold*: a smooth algebraic variety having a torus action with an open dense orbit.
  - *quasitoric manifold*: a topological version of toric variety
- They produce rich interaction

$$\text{Geometry} \Leftrightarrow \text{Combinatorics}$$

**GKM theory**

provides a uniform (but weak) machinery to deal with all those classes

# Motivation

GKM-theory captures less information:

| manifold | combinatorics | relation to GKM |
|---|---|---|
| flag mfd | Coxeter group | Hasse diagram |
| quasitoric mfd | simple polytope | 1-skelton |
| GKM mfd | GKM-graph | |

**Question**

What is dropped when we see special mfds merely as GKM-mfds ?
How can we strengthen the theory with some additional assumption ?

# Motivation

GKM-theory captures less information:

| manifold | combinatorics | relation to GKM |
|---|---|---|
| flag mfd | Coxeter group | Hasse diagram |
| quasitoric mfd | simple polytope | 1-skelton |
| GKM mfd | GKM-graph | |

**Question**

What is dropped when we see special mfds merely as GKM-mfds ?
How can we strengthen the theory with some additional assumption ?

# Goal

## General Goal

Extend combinatorial machinery for flag mfd and toric mfd to GKM mfd

The study of

$$\text{Geometry} \Leftrightarrow \text{Combinatorics}$$

- for flag mfd is known as *Schubert calculus*
- for quasitoric mfd is known as *Toric topology*

Now we briefly recall them.

# Schubert calculus in one slide

- Let $T$ be a maximal torus of a simple complex Lie group $G$
- Let $T \subset B \subset P \subset G$ be a Borel and a parabolic subgroup
- Let $W(G)$ and $W(P)$ be the Weyl groups of $G$ and $P$
- The flag mfd $G/P$ admits a $T$-action with the fixed pts set identified with $W(G)/W(P)$
- $G/P$ has a $T$-equivariant cell decomposition (by Chevalley)

$$G/P = \bigcup_{w \in W(G)/W(P)} BwP/P$$

- The closure $\sigma_w := \overline{BwP/P}$ is called the *Schubert variety*
- It relates the combinatorics on $W(G)/W(P)$ to geometry:
  - $w \leq v$ by the Bruhat order on $W(G)/W(P) \Leftrightarrow \overline{\sigma_w} \supset \sigma_v$
  - $\dim \sigma_w = 2l(w)$, two times the *length* of $w \in W(G)/W(P)$
  - $H^*(G/P) = \bigoplus_{w \in W(G)/W(P)} \mathbb{Z}\langle \sigma_w \rangle$
  - This induces Sperner property, ring structure, etc. on $W(G)/W(P)$

# Schubert calculus

For example, the complex Grassmannian of $\mathbb{C}^2$'s in $\mathbb{C}^4$

$$Gr_2(4;\mathbb{C}) := SL_4(\mathbb{C})/P, \text{ where } P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

- $W(SL_4(\mathbb{C}))/W(P)$ is *Young diagrams* contained in $2 \times 2$-square.
- $H^*(Gr_2(4;\mathbb{C})) = \mathbb{Z}\langle 1, \ydiagram{1}, \ydiagram{2}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2} \rangle$.
- Ring structure on diagrams: Ex. $\ydiagram{1} \cdot \ydiagram{1} = \ydiagram{1,1} + \ydiagram{2}$
  satisfying Poincare duality and strong Lefchetz property
- Weyl group action on $\mathbb{Z}\langle 1, \ydiagram{1}, \ydiagram{2}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2} \rangle$
  This determines the ring structure
  (through divided difference operators)

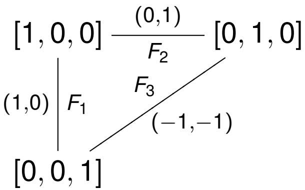
# Toric topology in one slide

- $2n$-dim mfd $M$ is a *quasitoric mfd* if
  - $T^n \curvearrowright M^{2n}$: locally standard effective action
  - The orbit space $M/T$ is a polytope $P$
- Each facet $F_i$ of $P$ corresponds to a $T$-stable submfd $N_i$ of codimension 2 (called *characteristic sub-mfd*)
- $H_T^*(M)$ is the Stanley-Reisner algebra $\mathbb{Z}[N_i]/SR$
- Let $c_i \in H^2(BT) = \mathbb{Z}^n$ be the first Chern class of $N_i \hookrightarrow M$ ( Facet $F_i \mapsto \mathbb{Z}^n$-vector $c_i$ )
- There is a categorical equivalence:
  { polytope & $\mathbb{Z}^n$-vector on facets (with certain conditions)}
  $\Leftrightarrow$ { quasitoric mfd (modulo $T$-equiv homeo)}

# Toric topology

For example, $M = \mathbb{C}P^2$ with the standard action of $T^2 = S^1 \times S^1$

- $M/T$ is the triangle




- $H_T^*(M) = \mathbb{Z}[F_1, F_2, F_3]/(F_1F_2F_3)$
- $H^*(M) = \mathbb{Z}[F_1, F_2, F_3]/(F_1F_2F_3, F_1 - F_2, F_1 - F_3)$
- Facets correspond to $T$-stable sub-mfds
  Ex. $F_1 \Leftrightarrow \mathbb{C}P^1 = \{[*, 0, *]\} \subset \mathbb{C}P^2$
  This determines the ring structure via intersection.

# What makes difference

We focus on the following properties:

- $G$-action (extending $T$-action) of flag mfd
  ($\Rightarrow$ Weyl group action on the Schubert varieties
  $\Rightarrow$ Ring structure on (the free $\mathbb{Z}$-module over) the poset $W(G)/W(P)$)
- $T$-stable sub-mfds of quasitoric mfd
  ($\Rightarrow$ Ring structure in the face ring)

since they determine the ring structures, which is the key ingredient for

$$\text{Geometry} \Leftrightarrow \text{Combinatorics}$$

## Goal Today

Export those to enhance GKM-theory

# GKM-theory

# Equivariant cohomology

When a group $G$ acts on a manifold $M$,
we can define the *$G$-equivariant cohomology* of $M$:

- $M \hookrightarrow EG \times_G M \to BG$: Borel construction
- $H_G^*(M;\mathbb{R}) := H^*(EG \times_G M;\mathbb{R})$: the $G$-equivariant cohomology

It captures the $G$-equivariant topology of $M$:

- Ex. if $G$ acts freely on $M$, then $H_G^*(M) = H^*(M/G)$
- Ex. if $G$ acts trivially on $M$, then $H_G^*(M) = H^*(BG) \otimes H^*(M)$
- In particular, $H_G^*(pt) = H^*(BG) \cong \mathbb{R}[t_1, \ldots, t_n]^W$

If $H_G^*(M)$ is free over $H_G^*(pt)$, the ordinary cohomology can be recovered

$$H^*(M) = \mathbb{R} \otimes_{H_G^*(pt)} H_G^*(M)$$

# GKM (Goresky-Kottwitz-MacPherson) manifold

Let $M$ be a compact manifold with an effective $T$-action

## Definition (Goresky-Kottwitz-MacPherson)

$M$ is GKM $\Leftrightarrow$

- the fixed-pts set $M^T$ and 1-dim orbits $M^1$ form a graph

$$(M^1 := \{m \in M \mid \dim(Tm) = 1\})$$

- Serre-SS for $M \hookrightarrow E \times_T M \to BT$ degenerates at $E_2$ (said to be *equivariantly formal*)

## Example

- flag mfds
- (quasi)toric manifolds
- torus manifolds with $H^{odd} = 0$

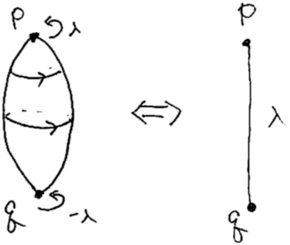
# GKM graph

For a GKM-manifold we associate the *GKM graph*:

The GKM graph $\mathcal{G}$ consists of

- $T$-fixed points $M^T$ as *vertices*
- 1-dim orbits $M^1$ as *edges*
- weight $\lambda_{p,q} \in H^2(BT)$ of the 1-dim rep on $T_pM$ as *labels*

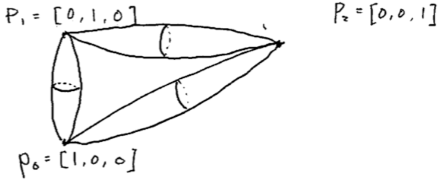
Ex. $\mathbb{C}P^1 = S^2$ with $\lambda$-times the standard $T^1$-action



# Figure of GKM manifold

- Let $M = \mathbb{C}P^2$
- $T^2 = \{((1, e^{t_1}, e^{t_2})\} \curvearrowright \mathbb{C}P^2$ by multiplication
- $M^T = \{p_0 = [1,0,0], p_1 = [0,1,0], p_2 = [0,0,1]\}$

1-skeleton of this GKM-manifold looks like:




- $H^*(BT) = \mathbb{Q}[t_1, t_2]$
- $T_{p_0}(M) = \mathbb{C}_{t_1} \oplus \mathbb{C}_{t_2}$
- $T_{p_1}(M) = \mathbb{C}_{-t_1} \oplus \mathbb{C}_{t_2 - t_1}$
- $T_{p_2}(M) = \mathbb{C}_{t_1 - t_2} \oplus \mathbb{C}_{-t_2}$

# $H_T^*(M)$: GKM description

The inclusion of fixed points

$$M^T \xrightarrow{i} M$$

induces an injection on cohomology called *the localization map*

$$H_T^*(M) \xrightarrow{i^*} H_T^*\left(M^T\right) \cong \bigoplus_{p \in M^T} H_T^*(pt)$$

The image of $i^*$ is determined by:

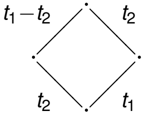
**Theorem (Goresky-Kottwitz-MacPherson)**

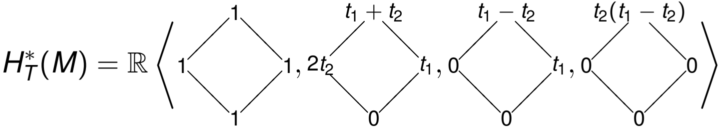
$$H_T^*(M;\mathbb{R}) \cong \mathrm{Im}(i^*) \cong H^*(\mathcal{G}) \subset \bigoplus_{p \in M^T} H^*(BT)$$

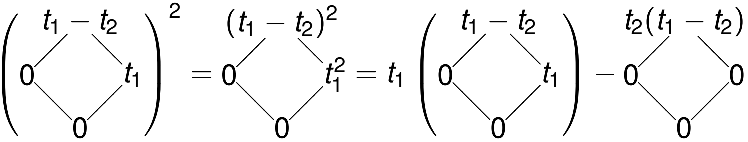
$$H^*(\mathcal{G}) := \{ \bigoplus_{p \in M^T} h_p(t) \mid h_p(t) \in H^*(BT), \quad \lambda_{p,q} | h_p(t) - h_q(t) \text{ if } p \xrightarrow{\lambda_{p,q}} q \}$$

# Example

$T^2 \curvearrowright M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, whose GKM-graph is



What happens if the $T$-action extends to a $G$-action

# GKM $G$-manifold

Recall that a flag mfd $G/P$, considered as the right quotient, has the left $G$-action by the multiplication.
We generalize this situation.

**Definition**

$M$ is a GKM $G$-mfd $\Leftrightarrow$
A Lie group $G$ acts on a manifold $M$ in such a way that
$M$ becomes a GKM-mfd with respect to the action restricted to $T$.

**Example**

- Homogeneous spaces $G/H$ with the multiplication action of $K$ $(H \subset K \subset G)$
- Kuroki determined when a quasitoric manifold is a GKM $G$-manifold

# Weyl group action on GKM graph

A GKM graph $\mathcal{G}$ is said to have a $W$-symmetry if

1. $W := W(G)$ acts on the vertices $M^T$
2. There must be an edge $p \overset{\beta}{\text{——}} s_\beta p$ for any reflection $s_\beta \in W$
3. If there is an edge

$$p \overset{\lambda}{\text{————}} q\ ,$$

then so is

$$wp \overset{w(\lambda)}{\text{————}} wq$$

**Theorem (K)**

The GKM-graph of a GKM $G$-manifold has a $W$-symmetry.

Note: Combined with Kuroki's result, up to dim 6,
$W$-symmetry on $\mathcal{G} \Leftrightarrow \exists G$-action on $M$

# Quotient GKM graph

If $\mathcal{G}$ has $W$-symmetry, we can define the quotient graph $\mathcal{G}/W$ as

- $M^T/W$ as vertices: we pick a set of representatives $\{p_1, \ldots, p_N\}$
- $p_i \xrightarrow{(w,\lambda)} p_j \overset{\text{def}}{\Longleftrightarrow} p_i \xrightarrow{\ \lambda\ } wp_j$ for some $w \in W/W_j$, where $W_j$ is the Weyl group of the isotropy group

$$P_j = \{g \in G \mid gp_j = p_j\}$$

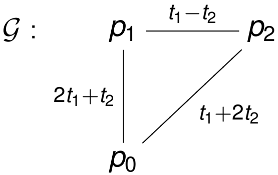
## Corollary

$H^*_G(M) \cong H^*(\mathcal{G}/W) \subset \bigoplus_{p_i \in M^T/W} H^*(BT)$

$$H^*(\mathcal{G}/W) := \{ \bigoplus_{p_i \in M^T/W} h_{p_i}(t) \mid h_{p_i}(t) \in H^*(BT)^{W_i},$$

$$\lambda | h_{p_i}(t) - w^{-1}(h_{p_j}(t)) \text{ if } p_i \xrightarrow{(w,\lambda)} p_j \}$$

# Example

- $T^2 = \{((t_1 t_2)^{-1}, t_1, t_2)\} \subset G = S(U(2) \times U(1)) \curvearrowright \mathbb{C}P^2$
- $(\mathbb{C}P^2)^T = \{p_0 = [1,0,0], p_1 = [0,1,0], p_2 = [0,0,1]\}$
- 

- $W = \langle s \rangle$, where $s(t_1) = -t_1 - t_2, \quad s(t_2) = t_2$
- $s(p_0) = p_1, s(p_2) = p_2$

# Example

The quotient graph is

$$\mathcal{G}/W: \ p_1 \overset{t_1 - t_2}{\text{———————}} p_2$$

Define classes in $H^*(\mathcal{G}/W)$ as

$\overline{x} = \ (2t_1 + t_2) - (3t_2)$ ,

$\overline{y} = \ (0) - (t_2 - t_1)(t_1 + 2t_2)$ , $\overline{z} = \ (t_2 - t_1) - (0)$ ,

$$\Rightarrow H^*_G(M) = H^*(\mathcal{G}/W) = \frac{\mathbb{Q}[\overline{x}, \overline{y}, \overline{z}]}{(yz)}$$

# Stable sub-mfds for GKM mfd

# Connection

There is additional structure on GKM-mfd called the *connection*:

- Let $\mathrm{star} p$ be the edges connected to $p \in M^T$
- each edge $p \overset{e}{\longrightarrow} q$ corresponds to a 2-sphere $X_e$
- the restriction of $TM$ to $X_e$ splits into

$$\bigoplus_{e' \in \mathrm{star}(p)} L_{e'},$$

  where the restriction of $L_{e'}$ at $p$ is isomorphic to $T_p X_{e'}$.
- For each edge $e \in E$, the *connection*

$$\theta_e : \mathrm{star}(p) \to \mathrm{star}(q)$$

  is the bijection which assigns to $e' \in \mathrm{star}(p)$ the edge corresponding to $L_{e'}|_q$.

# Connection on GKM-graph

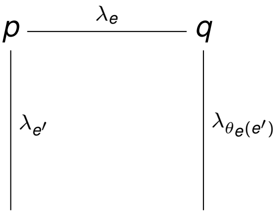
Combinatorially,
the connection is a set of bijections

$$\theta_e : \mathrm{star}(p) \to \mathrm{star}(q) \qquad (e \in E)$$

which satisfies

$$\lambda_e \mid \lambda_{\theta_e(e')} - \lambda_{e'}.$$

That is




$\Rightarrow \lambda_e$ divides $\lambda_{\theta_e(e')} - \lambda_{e'}$

# Characteristic sub-graph

Recall that characteristic sub-mfds ($T$-stable codim-2 sub-mfds) in a quasitoric mfd plays crucial role.
We define a graph analogy for them:

**Definition**

A sub-graph $\Lambda \subset \mathcal{G}$ is said to be *closed under the connection* $\theta$ when $\theta_e(e') \in \Lambda$ for $\forall e, e' \in \Lambda$.

Since $\theta_e$ is bijection for all $e \in E$, $\Lambda$ is always a regular graph. Let $\mathcal{G}_k$ be the set of all the $k$-valent subgraphs which are closed under the connection. For $\Lambda \in \Gamma_k$, we obtain the class $\lambda \in H^{2m-2k}(\mathcal{G})$ which is supported on $\Lambda$ by

$$N(\Lambda)_p = \begin{cases} \prod_{e \in \mathrm{star} p \setminus \Lambda} \lambda_e & (p \in \Lambda) \\ 0 & (p \notin \Lambda) \end{cases}.$$

# Characteristic sub-graph

**Theorem (K)**

For a GKM $G$-mfd,

$$w(\theta_e(e')) = \theta_{w(e)}(w(e')) \qquad \forall e, e' \in E, w \in W$$

and hence $W$ acts on the set $\mathcal{G}_k$.
Moreover, $W$ acts on the set $\{\pm N(\Lambda) \in H^{2m-2k}(\mathcal{G}) \mid \Lambda \in \Gamma_k\}$ as a signed permutation.

**Corollary (cf. Masuda, Wiemeler)**

If a quasitoric mfd admits $G$-action, then $G$ should be of classical type.

# Operators on $H^*(\mathcal{G})$

# Divided difference operator for $G/P$

For a root $\beta$, the (left) divided difference operator

$$\partial_\beta : H_T^*(G/P) \to H_T^{*-2}(G/P)$$

is defined as follows:
Let $P_\beta \subset G$ be the minimal parabolic corresponding to a root $\beta$,
(so that $P_\beta/T \cong \mathbb{C}P^1$ )
Consider the fiber bundle

$$P_\beta/T \hookrightarrow ET \times_T P_\beta \times_T G/P \xrightarrow{\pi} ET \times_T G/P,$$

where $\pi(e, g, m) = (eg, m)$.

## Definition

$$\partial_{s_\beta}(c) = \pi_*\pi^*(c)$$

# Divided difference operator

$\partial_\beta$'s are compatible with the poset structure on $W(G)/W(P)$:

$$\partial_{\alpha_i}\sigma_w = \begin{cases} \sigma_{s_i w} & (l(w) > l(s_i w)) \\ 0 & (l(w) < l(s_i w)) \end{cases}$$

and also related to the $W(G)$-action by

$$\partial_{\alpha_i} = (1 - s_i)/\alpha_i, \qquad \alpha_i : \text{ simple root}$$

Define

$$\partial_w = \partial_{s_{i_1}} \circ \partial_{s_{i_2}} \circ \cdots \circ \partial_{s_{i_{l(w)}}} \qquad \text{for } w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$$

then all the Schubert classes are obtained by

$$\sigma_w = \partial_{ww_0}\sigma_{w_0}, \qquad \sigma_{w_0} : \text{ the class of the base point}$$

The product in $H_T^*(G/P)$ is computed as

$$\alpha_i \sigma_w = \sigma_{s_i w} - s_i \sigma_{s_i w}.$$

# Divided difference on GKM-graph

We can define an analogy of $\partial_\beta$ purely combinatorially on $H^*(\mathcal{G})$:

**Proposition**

If a GKM-graph has a $W$-symmetry

$$\partial_\beta(h)_p = \frac{h_p - s_\beta(h_{s_\beta p})}{\beta}$$

is well-defined.
(and hence $H^*(\mathcal{G})$ admits an action of the nil-Hecke ring of $W$)

# Divided difference on GKM-graph

## Corollary

Let $w_0 \in W$ be the longest element. The *Becker-Gottlieb transfer* is identified with

$$\begin{aligned} \tau : H^*(\mathcal{G}) &\to H^*(\mathcal{G}/W) \\ h &\mapsto \partial_{w_0}(d\mathbf{1} \cdot h), \end{aligned}$$

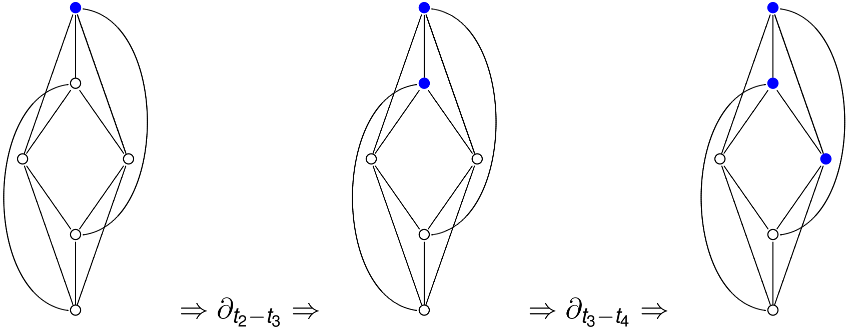
where $d = \prod_{\beta \in \Pi^+} \beta$.
Moreover, the *Brumfiel-Madsen formula*

$$\tau(h) = \sum_{w \in W} w(h)$$

holds and $\frac{1}{|W|}\tau$ is a left inverse to the inclusion $\iota : H^*_W(\Gamma) \hookrightarrow H^*(\Gamma)$.

# Example: GKM-description of $\partial_\beta$

Ex. $T^4 \curvearrowright Gr_2(\mathbb{C}^4) = U(4)/U(2) \times U(2)$




black circle: the class vanish at the point
blue circles: the class doesn't vanish at the point

# Summary

- GKM-theory deals with a wide class of mfds with $T$-action
- We extended GKM-theory when there is a Lie group action
  - For GKM $G$-mfd, the GKM-graph admits a symmetry of the Weyl group $W$ (as in the case of flag mfd)
  - We defined an analogy for characteristic submfds and $W$ acts on them as permutation (as in the case of quasitoric mfd)
  - We redefined Becker-Gottlieb transfer and rediscovered Brumfiel-Madsen formula in terms of GKM-graph

Future work:

- Classify GKM $G$-mfds
- Find sufficient condition for an existence of $G$-action in terms of GKM-graph
- Construct an “inverse” to GKM-graph: $\mathcal{G} \longrightarrow M_{\mathcal{G}}$

# Thank you very much

Thank you very much
for your attention !