Representing the diagonal as the zero locus in a flag manifold

Shizuo Kaji
Kyushu Univ.

Topology Seminar
29 Jun. 2018, Kyushu University

Introduction

- $f : N \to \mathbb{C}$: a smooth function on a manifold N
- having $0 \in \mathbb{C}$ as a regular value
- Then, $M := f^{-1}(0) \subset N$ is a sub-manifold

Naive Question

When can a given sub-manifold $M \subset N$ of codimension $2m$ be realised as the zero locus of a function $N \to \mathbb{C}^m$, or more generally, of a section of a rank m complex vector bundle $V \to N$?

A point in S^2

Example

- $\{pt\} \subset S^2 = \mathbb{C}P^1$
- $\gamma^* \to \mathbb{C}P^1$: the dual of the tautological bundle

$$\gamma = \{([x, y], (cx, cy)) \mid x, y, c \in \mathbb{C}\}.$$

- Define its section s by

$$s([x, y]) : (cx, cy) \mapsto cx,$$

whose zero locus is the point $\{[0, y]\} \subset \mathbb{C}P^1$.

The problem

- N: a closed oriented $2n$-manifold
- $M \subset N$: a sub-manifold of codimension $2m$

We say M is *representable* if

- $\exists \gamma \to N$: a complex m-bundle
- $\exists s : N \to \gamma$: a smooth generic section (i.e., transversal to the zero section)
- $Z(s) = M$ where $Z(s) := \{x \in N \mid s(x) = 0\}$ is the zero locus

Main question: When and how can M be represented?

Remark

We can consider variants of this definition by replacing complex bundles with real or oriented bundles.

The class of a sub-manifold

- $\iota : M \subset N$: an oriented embedding defines a cohomology class via
- $\iota_! : H^*(M) \to H^{*+\dim(N)-\dim(M)}(N)$: the Gysin map defined by
- $H^*(M) \xrightarrow{PD} H_{\dim(M)-*}(M) \xrightarrow{\iota_*} H_{\dim(M)-*}(N) \xrightarrow{PD} H^{*+\dim(N)-\dim(M)}(N)$
- $[M] := \iota_!(1) \in H^{\dim(N)-\dim(M)}(N)$: the class of M

Proposition (or definition?)

For a complex m-bundle γ over an n-manifold N with $2m \leq n$, the class of the zero locus of its generic section s is equal to the top Chern class:

$$[Z(s)] = c_m(\gamma) \in H^{2m}(N).$$

A point in S^{2n}

Proposition

Let $N = S^{2n}$. A point (resp. the diagonal) is representable iff $n = 1$.

This is an easy consequence of the following theorem:

Theorem (Bott)

The top Chern class c_n of any complex n-bundle on S^{2n} is divisible by $(n-1)!$.

In fact, this is the model case of the arguments which we see today.

Outline

- The classes of a point and the diagonal
- Flag manifold
- Type-A case
- Other types

Point and Diagonal classes

We focus on the representability of

- $\{pt\} \subset N$: a point
  ($[pt] \in H^*(N)$ is the generator of the top degree cohomology)
- $\Delta(N) \subset N \times N$: the diagonal

The problem has been studied by Fulton, Pragacz, Srinivas, and Pati.

These two extreme cases look rather special, but they are actually interesting instances as we see later.

Diagonal $\Rightarrow$ point

Assume that the diagonal is representable, i.e.,

- $2n = \dim(N)$
- $\gamma \to N \times N$: a complex n-bundle
- $s : N \times N \to \gamma$: a generic section
- $Z(s) = \Delta(N)$

Then, we can pullback everything along the inclusion to the first factor

$$i_1 : N \hookrightarrow N \times N, \quad (i_1(x) = (x, pt))$$

to represent a point

- $i_1^*(\gamma) \to N$: a complex n-bundle
- $i_1^*(s) : N \to i_1^*(\gamma)$: a generic section
- $Z(i_1^*(s)) = \{pt\}$

In particular, Δ is representable $\Rightarrow$ so is $\{pt\}$.

Flag manifold

Flag manifold

notation	typical example
G: simple complex Lie group	$SL_n(\mathbb{C})$
B: Borel subgroup	upper triangular matrices
P: Parabolic subgroup	upper triangular block matrices
T: maximal torus	diagonal matrices
G/P: flag manifold	$Gr_m(\mathbb{C}^n) = SL_n(\mathbb{C})/P_m$
$W(G)$: Weyl group of G	S_n: symmetric group
$W(P)$: Weyl group of P	$S_m \times S_{n-m}$: symmetric group

Note that for the maximal compact subgroup $K \subset G$,

$$G/P \simeq K/K \cap P \qquad \text{(in particular, } G/B \simeq K/T\text{)}$$

For example, $SL(k)/B \simeq SU(k)/T$.
We frequently switch between these notations.

Equivariant point ⇒ diagonal

By group induction, we have

$$H^*_G(X \times G/P) = H^*_P(X),$$

for any G-space X and $P \subset G$.
Then, we have

$$\begin{array}{ccccc} H^*_P(pt) & = & H^*_G(G/P) & \longrightarrow & H^*(G/P) \\ \downarrow i_! & & \downarrow \Delta_! & & \downarrow \Delta_! \\ H^*_P(G/P) & = & H^*_G(G/P \times G/P) & \longrightarrow & H^*(G/P \times G/P) \end{array}$$

This means, the P-equivariant class of a point $i_!([pt]) \in H^*_P(G/P)$ gives rise to the class of the diagonal $[\Delta(G/P)] \in H^*(G/P \times G/P)$.

Remark: Schubert classes

G/B has the *Bruhat cell decomposition*

$$G/B = \bigcup_{w \in W(G)} BwB/B.$$

where $BwB/B \simeq \mathbb{C}^{l(w)}$. The closure of the cell BwB/B is the Schubert variety X_w. They can be non-smooth but they admit nice resolutions and define cohomology classes.

Theorem (Chevalley)

$$H^*(G/B) \simeq \bigoplus_{w \in W(G)} \mathbb{Z}\langle [X_w] \rangle$$

$$H^*_T(G/B) \simeq \bigoplus_{w \in W(G)} H^*(BT)\langle [X_w] \rangle$$

Remark: Schubert classes

All the Schubert classes are obtained from the class of a point $[pt] \in H^{\dim(G/B)}(G/B)$ by application of the divided difference operators:

$$[X_w] = \delta_w([pt]).$$

In this sense, $[pt]$ generates $H^*(G/B)$.

Similarly, the T-equivariant cohomology $H_T^*(G/B)$ is generated by the class of the G-equivariant diagonal ($=$ T-equivariant point).

This is why we care the point and the diagonal.

Type-A case

Type-A flag manifold

When $G = SL(k)$, the corresponding flag manifolds are of the form

$$SL(k)/P = \{0 \subsetneq V_{i_1} \subsetneq V_{i_2} \subsetneq \cdots \subsetneq V_{i_m} \subset \mathbb{C}^k \mid \dim_{\mathbb{C}}(V_{i_m}) = i_m\},$$

where $(i_1, i_2, \ldots, i_m)$ depends on P.

Theorem (Fulton)
The diagonal in G/P is representable for any P when $G = SL(k)$.

Proof of Fulton's theorem

Consider the universal flag bundle

$$G/P \hookrightarrow BP \to BG$$

and the pullback diagram

$$\begin{array}{ccc} G/P \times G/P & = & G/P \times G/P \\ \downarrow & & \downarrow \\ BP \times_{BG} BP & \longrightarrow & BP \times BP \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\Delta} & BG \times BG. \end{array}$$

A bundle over $BP \times BP$ and its section s on the closed subset $BP \times_{BG} BP$ with $Z(s) = \Delta(BP)$ will be constructed.

Let $\pi_1, \pi_2 : BP \times BP \to BP$ be the projections.
As $G/P \to BP \to BG$ is the universal flag bundle, we have the tautological sequence of bundles on BP:

$$S_1 \xrightarrow{\iota} S_2 \xrightarrow{\iota} \cdots \xrightarrow{\iota} S_m \xrightarrow{\iota} \gamma_{l+1} \xrightarrow{q} Q_1 \xrightarrow{q} \cdots \xrightarrow{q} Q_m,$$

where γ_{l+1} is the pullback of the universal bundle over BG. Consider the bundle map

$$\phi : \bigoplus_i \hom(\pi_1^*(S_i), \pi_2^*(Q_i)) \to \bigoplus_i \hom(\pi_1^*(S_i), \pi_2^*(Q_{i+1}))$$

over $BP \times BP$ defined by $\bigoplus_i h_i \mapsto \bigoplus_i (q \circ h_i - h_{i+1} \circ \iota)$.
The bundle $\ker \phi \to BP \times BP$ admits a restricted section $s_\phi : BP \times_{BG} BP \to \ker \phi$ defined by the tautological map $\pi_1^*(S_i) \to \pi_1^*(\gamma_{l+1}) = \pi_2^*(\gamma_{l+1}) \to \pi_2^*(Q_i)$, which vanishes exactly along the diagonal.

Flag bundle

Fulton's argument can be slightly generalised to give

Proposition

If the diagonal (resp. a point) is representable for N, then so is for the flag bundle $Fl(V)$ with any bundle $V \to N$.

In particular, when $P_1 \subset G$ is of type A and $P_1 \supset P_2$, from the following fibre bundle

$$P_1/P_2 \hookrightarrow G/P_2 \xrightarrow{p} G/P_1$$

we see the representability for G/P_2 follows from that of G/P_1. (This kind of argument is generally called *parabolic induction*.)

Other types

(Non-)Representability

Theorem

For a simple complex Lie group G, the diagonal (resp. a point) in G/B is representable iff G is of type A or C.

Type $C \Rightarrow$ representable is proved by parabolic induction

$$P(S_j^{\perp}/S_j) = P_{C_j}/P_{C_{j-1}} \hookrightarrow G_{C_k}/P_{C_{j-1}} \to G_{C_k}/P_{C_j},$$

where $0 \subset S_1 \subset \cdots \subset S_k = S_k^{\perp} \subset \cdots \subset S_1^{\perp} \subset \mathbb{C}^{2k}$ is the tautological isotropic flag.

Theorem

For a simple complex Lie group G of exceptional type, a point in G/P for any P is not representable.

Proving non-representability

We will check if there exists $\gamma \in \mathrm{Vect}^n_{\mathbb{C}}(N)$ such that $[M] = c_n(\gamma) \in H^{2n}(N)$.
In particular for $M = \{pt\}$, we will examine if

$$c_n : \mathrm{Vect}^n_{\mathbb{C}}(N) \to H^{2n}(N)$$

is surjective.
Actually, as Chern classes are stable, we can think of them as

$$c_n : K^0(N) \to H^{2n}(N).$$

And $K^0(N)$ is much easier to work with than $\mathrm{Vect}_{\mathbb{C}}(N)$!

K-theory of $G/B \simeq K/T$

We can assume that G is simply-connected.

For $\lambda \in \hom(T, \mathbb{C}^*) \simeq H^2(BT)$, define a line bundle L_λ by

$$K \times_T \mathbb{C}_\lambda \to K/T.$$

We have

$$H^2(BT) \to K^0(K/T)$$
$$\lambda \mapsto L_\lambda,$$

which extends to a surjection (with the kernel generated by $H^+(BT)^W$)

$$H^*(BT) \to K^0(K/T).$$

This means every bundle stably splits into a sum of line bundles.

Torsion index

We now know that the image of

$$c_n : \mathrm{Vect}^n_{\mathbb{C}}(K/T) \to H^{2n}(K/T)$$

is same as the image of c_n of sums of line bundles (of various ranks). So it is in the sub-algebra generated by $H^2(K/T) \simeq H^2(BT)$.

The minimum integer τ such that $\tau \cdot [pt]$ is in the sub-algebra generated by $H^2(BT)$ is called the *torsion index* of K. The torsion indexes of simple, simply-connected Lie groups are determined by Grothendieck, Demazure, Marlin, Tits, and Totaro.
Especially, $\tau = 1$ iff K is of type A or C.

This shows the only if part:
$\{pt\} \in K/T$ is representable $\Rightarrow K$ is of type A or C.

Diagonal in $Sp(l)/T$

Define the following bundle over $BT \times BT$:

$$\xi := \bigoplus_{1 \le i \le j \le l} L_{x_i} \hat{\otimes} L_{y_j} \bigoplus_{1 \le i < j \le l} L_{x_i} \hat{\otimes} L_{y_j}^*,$$

where $x_i = x_i \otimes 1, y_i = 1 \otimes x_i \in H^2(BT \times BT)$ are generators. Consider

$$\iota : Sp(l)/T \times Sp(l)/T \to ESp(l) \times_T Sp(l)/T \to BT \times BT$$

Then, we can show by localisation to the fixed points that

Proposition

$c_k(\iota^* \xi) = [\Delta] \in H^{2k}(Sp(l)/T \times Sp(l)/T)$

Partial flags G/P

When $P \neq B$, we only have partial results. First, note that

$$K^0(G/P) = K^0(G/B)^{W(P)}, \quad H^*(G/P) = H^*(G/B)^{W(P)}.$$

Let $\tau_{G/P}$ be the least integer such that $\tau_{G/P} \cdot [pt]$ is in the image of

$$H^*(BT)^{W(P)} \to H^*(G/P).$$

By the bundle $P/B \to G/B \to G/P$, we see $\tau_G = \tau_{G/B} \leq \tau_{P/B} \cdot \tau_{G/P}$
People have calculated

$$\tau_{G_2} = 2, \tau_{F_4} = 6, \tau_{E_6} = 6, \tau_{E_7} = 12, \tau_{E_8} = 2880$$

so for any parabolic $\tau_{G/B} > \tau_{P/B}$ for exceptional groups. Therefore, $t_{G/P} > 1$ and pt is not representable in G/P when G is exceptional.

G/P for classical types

The previous argument works for some G/P with G of classical types but not all.
We can also use generalised Fulton's theorem to see

Proposition

A point in the *Lagrangian Grassmann* for the quadratic vector space $\mathbb{C}^{2k}$

$$Lag(\mathbb{C}^{2k}) \simeq SO(2k)/U(k) = SO(2k-1)/U(k-1)$$

is representable iff $k \leq 3$.

Proof.
Apply Fulton's theorem to the bundle

$$U(k)/T \hookrightarrow SO(2k)/T \to SO(2k)/U(k).$$

□

Lagrangian Grassmann

In all cases so far, representability of a point and the diagonal are equivalent.
Of course, this does not hold in general.

Proposition

A point in $Lag_\omega(\mathbb{C}^{2k}) \simeq Sp(k)/U(k)$, the Lagrangian Grassmann for the symplectic vector space $\mathbb{C}^{2k}$ is representable by

$$\xi := \bigoplus_i L_{x_i} \bigoplus_{i<j} L_{x_i+x_j}.$$

Remark: $Sp(k)/U(k) \simeq_{p:odd} SO(2k+1)/U(k)$. So 2-torsion plays an important role!

Theorem

When $k \equiv 2 \mod 4$, the diagonal in $Sp(k)/U(k)$ is not representable.

Let $2n = \dim(Sp(k)/U(k)) = k(k+1)$.

Lemma

$$H^*(Sp(k)/U(k)) \simeq \frac{\mathbb{Z}[c_1, c_2, \ldots, c_k]}{(\mathbb{Z}[q_1, q_2, \ldots, q_k])^+},$$

where c_i (resp. q_i) are elementary symmetric functions of x_j (resp. x_j^2). Here, $H^*(BT) = \mathbb{Z}[x_1, \ldots, x_k]$. In particular,

$$u_n = \prod_i x_i \prod_{i<j} x_i + x_j = \prod_{i=1}^{l} c_i$$

$$u_{n-1} = \prod_{i=2}^{k} c_i$$

$$u_1 = c_1$$

are the generators of
$H^{2n}(Sp(k)/U(k)), H^{2n-2}(Sp(k)/U(k)), H^2(Sp(k)/U(k))$.

$N = Sp(k)/U(k)$

Lemma
Any bundle $V \to N$ representing $\{pt\} \subset N$ is spin.

Proof.
We show $c_n(V) = u_n$ imlies $c_1(V) \equiv w_2(V) = 0 \mod 2$.
Note $H^*(N; \mathbb{Z}/2\mathbb{Z}) \simeq H^*(N; \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$.
We have $Sq^2 u_{n-1} = (l-1)u_n = u_n$.
Set $c_1(V) = au_1$ and $c_{n-1}(V) = bu_{n-1}$.
By Wu's formula (here we use $n \equiv 1 \mod 2$)

$$bu_n \equiv Sq^2(c_{n-1}(V)) \equiv c_1(V)c_{n-1}(V) + c_n(V) \equiv (ab+1)u_n.$$

So $b(a+1) \equiv 1$, and hence, $a \equiv 0$. □

$N = Sp(k)/U(k)$

Assume that $\xi \to N \times N$ represents the diagonal Δ.
Consider the pull back of ξ along $\Delta : N \to N \times N$:

$$\Delta^*(\xi) \simeq TN = \bigoplus_i L_{2x_i} \bigoplus_{i<j} L_{x_i+x_j}.$$

On the other hand, its pullback along the inclusion to each factor $i_1, i_2 : N \to N \times N$ represents a point in N.
Since $i_1^* \otimes i_2^* : H^2(N \times N) \simeq H^2(N) \otimes H^2(N)$, we see

$$(k+1)u_1 = c_1(TN) = c_1(\Delta^*(\xi)) = \Delta^*(c_1(\xi)) = c_1(i_1^*(\xi)) + c_1(i_2^*(\xi)) \equiv 0.$$

This contradicts that k is even.

Conclusion and Future work

	$SL(k)/P$	$Spin(k)/B$	$Spin(k)/P$	$Sp(k)/B$	$Sp(k)/U(k)$	exceptional
pt	o	x	?	o	o	x
Δ	o	x	?	x	$?x$	x

Open problem: *Complete the table* by filling "?"
The remaining cases are

- $Spin(l)/P$ when P is of type B and D:
  In many cases, we can see pt is not representable by looking at the computation of the torsion index by Totaro.
- $Sp(l)/P$ when P is of type A and C

Thank you!