HOMOTOPY NILPOTENCY IN p-REGULAR LOOP SPACES

SHIZUO KAJI AND DAISUKE KISHIMOTO

Abstract. We consider the problem: how far from being homotopy commutative is a loop space having the homotopy type of the p-completion of a product of finite numbers of spheres? We determine the homotopy nilpotency of those loop spaces as an answer to this problem.

1. Introduction

For a prime p, we mean the p-localization and the p-completion in the sense of Bousfield and Kan [4], and denote them by $-_{(p)}$ and $-_p^\wedge$ respectively. We will always assume all spaces have the homotopy types of CW-complexes.

H-spaces have been of great interest in algebraic topology and extensively studied. Among other things, the study on the homotopy types of homotopy commutative H-spaces is very successful since we have celebrated Hubbuck's torus theorem [12]: homotopy commutative connected finite H-spaces are homotopy equivalent to tori. Its mod p analogues are obtained later in various forms by [2], [20], [18] and others. On the other hand, regarding homotopy commutativity, we can consider another problem to determine whether a given H-space is homotopy commutative or not. Now let G be a compact connected Lie group. Mislin [23] showed that the commutator map of G is of finite order in the group of the homotopy set $[G \times G, G]$. This implies that $G_{(p)}$ is homotopy commutative if p is large enough. Motivated by this, McGibbon [19] addressed the question:

Question 1.1. *For which prime p is the p-localized Lie group homotopy commutative?*

McGibbon [19] gave the following complete answer to this question when the Lie groups are simple, which also answers the case of simply connected Lie groups. Let G be a compact connected Lie group. It is classical that the rationalization of G has the homotopy type of the rationalization of $\prod_{i=1}^{l} S^{2n_i-1}$ with $n_1 \leq \ldots \leq n_l$. In this case, we say that G is of type $(n_1, \ldots, n_l)$.

Theorem 1.2 (McGibbon [19])**.** *Let G be a compact connected simple Lie group of type $(n_1, \ldots, n_l)$. Then we have:*

1. *$G_{(p)}$ is homotopy commutative if $p > 2n_l$.*
2. *$G_{(p)}$ is not homotopy commutative if $p < 2n_l$ except for $(G, p) = (\mathrm{Sp}(2), 3)$, $(G_2, 5)$.*

This result also holds if we replace the p-localization by the p-completion since it only deals with the p-torsion in the group of the homotopy set $[G \times G, G]$ as above (see [23]).

It is worth continuing the study on the multiplicative structures of p-completed Lie groups, or, more generally, p-complete loop spaces. Then we consider:

Question 1.3. *How far from being homotopy commutative is a given p-complete loop space?*

In order to study Question 1.3, we have to measure a *distance* from homotopy commutativity. In group theory, if we consider only nilpotent groups, the nilpotency class is the one which measures a *distance* from commutativity. Then we will adopt the homotopy nilpotency for our purpose which is defined as follows. Let X be a group-like space and let $\gamma_1 : X \times X \to X$ be the commutator map. Put $\gamma_n = \gamma_1 \circ (1 \times \gamma_{n-1})$ for $n > 1$. Then γ_n is the n-iterated commutator map of X. X is called homotopy nilpotent of class n if γ_n is null homotopic but γ_{n-1} is not (see [30]). In this case, we write $\mathrm{nil}X = n$. Then the homotopy nilpotency is the homotopy analogue of the nilpotency of groups. Note that $\mathrm{nil}X$ is normalized as $\mathrm{nil}X = 1$ if and only if X is homotopy commutative. It is obvious that, for homotopy nilpotent group-like spaces $X_1, \ldots, X_k$, we have

$$\mathrm{nil}(X_1 \times \cdots \times X_n) = \max\{\mathrm{nil}X_1, \ldots, \mathrm{nil}X_k\}. \tag{1}$$

Of course, not all p-complete loop spaces are homotopy nilpotent but the p-complete loop spaces which we will deal with are all homotopy nilpotent. Then our adoption is plausible.

Now we specify the class of p-complete loop spaces which we will deal with. Let X be a p-complete loop space. Since we regard Question 1.3 as a generalization of Question 1.1, we should impose a finiteness condition on X such that the mod p cohomology of X is finite. Those loop spaces are now called p-compact groups [7]. We start considering Question 1.3 for relatively large primes since the homotopy types of p-compact groups are simple as follows. Let X be a simply connected p-complete H-space having a finite mod p cohomology. As is seen above, the rationalization of X has the homotopy type of $\prod_{i=1}^{l} K(\mathbf{Q}_p^\wedge, 2n_i - 1)$ with $n_1 \leq \cdots \leq n_l$. In this case, we say that X is of type $(n_1, \ldots, n_l)$. Of course, this definition of types of p-complete H-spaces is consistent with those of Lie groups above. If X has the homotopy type of the p-completion of $\prod_{i=1}^{l} S^{2n_i-1}$, then we say that X is p-regular. Recall that Kumpel [17] generalized the classical result of Serre [27] as follows. If a simply connected p-complete H-space is of type $n_1, \ldots, n_l$ and $p \geq n_l - n_1 + 2$, then it is p-regular. Summarizing, the aim of this paper is to consider:

Question 1.4. *How large is the homotopy nilpotency of a given simply connected p-regular p-compact group ?*

Remark 1.5. *A continuation of Question 1.4 is considered by the second named author* [15]. *Actually, he considers the homotopy nilpotency of a p-localized* $\mathrm{SU}(n)$ *having the homotopy type of a product of spheres and sphere bundles over spheres. As in the following theorem, one can expect that* $\mathrm{nil}G_{(p)}$ *is a monotonic decreasing function in p for a fixed Lie group G in most cases. But it is shown in* [15] *that this is false.*

It is known that p-compact groups have maximal tori and Weyl groups [7], and, for each p-adic pseudoreflection group, there is a connected p-compact group, unique up to isomorphism, having the Weyl group isomorphic to it. We call connected simple p-compact groups which are not the p-completion of Lie groups by exotic p-compact groups (see [1]). Exotic p-compact groups are known to be simply connected and they correspond via Weyl groups to p-adic pseudoreflection groups in Clark and Ewing's list [5] except for five types when p is odd, and the Dwyer and Wilkerson's exotic loop space [6] when $p = 2$. Recently, Andersen, Grodal, Møller and Viruel [1] and Møller [24], [25] gave the classification of p-compact groups in such a way that each connected p-compact group is isomorphic to a product of the p-completion of a Lie group and exotic p-compact groups. Then, by (1), it is sufficient to consider the homotopy nilpotency of p-regular exotic p-compact groups and the p-completion of compact simply connected simple Lie groups which are p-regular in order to answer Question 1.4.

Let X be a simply connected p-compact group of type $(n_1, \ldots, n_l)$. Wilkerson [29] showed that X is not p-regular for $p < n_l$. Then, by combining with Kumpel's result [17] above, X is p-regular if and only if $p \geq n_l$. Then we can list up all p-regular simple p-compact groups and, in particular, five exceptional types of exotic p-compact groups are not p-regular. Now we state the results:

Theorem 1.6. *Let G be a compact, simply connected, simple Lie group of type $(n_1, \ldots, n_l)$. Then we have:*

(1) $\mathrm{nil}G_{(p)} = 3$ *if* $n_l \leq p \leq \frac{3}{2}n_l$ *for* $(G, p) = (F_4, 17), (E_6, 17), (E_8, 41), (E_8, 43), (\mathrm{SU}(2), 2)$.
(2) $\mathrm{nil}G_{(p)} = 2$ *if* $\frac{3}{2}n_l < p < 2n_l$ *or* (G, p) *is in the above exceptional case.*

We refer the number of the p-adic pseudoreflection group W in Clark and Ewing's list [5] to $N(X)$ if an exotic p-compact group X corresponds to W.

Theorem 1.7. *Let X be an exotic p-compact group of type $(n_1, \ldots, n_l)$. Then we have:*

(1) $\mathrm{nil}X = 1$ *if* $p > 2n_l$.
(2) $\mathrm{nil}X = 2$ *if* $n_l < p < 2n_l$ *except for* $(N(X), p) = (2\mathrm{b}, n_l + 1), (23, 11), (30, 31)$.
(3) $\mathrm{nil}X = 3$ *if* $(N(X), p)$ *is in the above exceptional case.*

The organization of this paper is as follows. In section 2, we will decompose the iterated commutator map γ_n above and, by using the classical result on the odd primary component of the homotopy groups of spheres, we will prove a result analogous to Theorem 1.6 and Theorem 1.7 in a more general setting but including some indeterminacy. In section 3, the above indeterminacy for Lie groups will be fixed in a case by case analysis. The case of the classical groups will be an easy consequence of the result of Bott [3]. We will show a cohomological criterion for a Samelson product being nontrivial. In the case of the exceptional Lie groups, by calculating the action of $\mathcal{P}^1$, we will use this criterion to fix the indeterminacy. In section 4, we also fix the indeterminacy of exotic p-compact groups analogously to the exceptional Lie groups using the above cohomological criterion.

2. Commutator map and Samelson products

We begin with the following easy lemma in which we normalize nilpotency of groups such that a group is nilpotent of class one if and only if it is abelian. Let K be a group generated by $x_1,\dots,x_l$. For $x,y\in K$, we write $x^y=yxy^{-1}$. For a positive integer n, we denote by K_n the subgroup of K generated by $[x_{i_1}^{\pm 1},[\cdots[x_{i_n}^{\pm 1},x_{i_{n+1}}^{\pm 1}]\cdots]]^y$ for all $1\le i_1,\dots,i_{n+1}\le l$ and $y\in K$, where $[-,-]$ is the commutator in K.

Lemma 2.1. *Let K be as above. Then, for any $y_1,\dots,y_{n+1}\in K$, the commutator $[y_1,[\cdots[y_n,y_{n+1}]\cdots]]$ belongs to K_n. In particular, K is nilpotent of class $<n$ if and only if $[x_{i_1}^{\pm 1},[\cdots[x_{i_n}^{\pm 1},x_{i_{n+1}}^{\pm 1}]\cdots]]=e$ for all $1\le i_1,\dots,i_{n+1}\le l$, where e is unity of K.*

Proof. For $x,y,z\in K$, we have

$$[x,yz]=[x,y][x,z]^y. \tag{2}$$

Let $x=x_{i_1}^{\pm 1}\cdots x_{i_a}^{\pm 1}$ and $y=x_{j_1}^{\pm 1}\cdots x_{j_b}^{\pm 1}$ for $1\le i_1,\dots,i_a,j_1,\dots,j_b\le l$. Then it follows that, for all $y=y_1\cdots y_a, z=z_1\cdots z_b\in K$,

$$\begin{aligned}
[y,z]&=[y,z_1][y,z_2\cdots z_b]^{z_1}\\
&=[y,z_1][y,z_2]^{z_1}[y,z_3\cdots z_b]^{z_1z_2}\\
&=\cdots\\
&=\prod_{i=1}^{b}[y,z_i]^{z_1\cdots z_{i-1}}\\
&=\prod_{i=1}^{b}[y_2\cdots y_a,z_i]^{z_1\cdots z_{i-1}y_1}[y_1,z_i]^{z_1\cdots z_{i-1}}\\
&=\prod_{i=1}^{b}[y_3\cdots y_a,z_i]^{z_1\cdots z_{i-1}y_1y_2}[y_2,z_i]^{z_1\cdots z_{i-1}y_1}[y_1,z_i]^{z_1\cdots z_{i-1}}\\
&=\cdots\\
&=\prod_{i=1}^{b}\prod_{j=1}^{a}[y_j,z_i]^{z_1\cdots z_{i-1}y_1\cdots y_{j-1}}.
\end{aligned}$$

Thus, by combining with the formula

$$[x,y^z]=[x^{z^{-1}},y]^z,$$

for $x,y,z\in K$, Lemma 2.1 follows from induction on n. □

Let X be a group-like space. We denote the composition of a map $\alpha:A\to X$ followed by the homotopy inverse of X by $-\alpha$. The (generalized) Samelson product of $\alpha:A\to X$ and $\beta:B\to X$, denoted $\langle\alpha,\beta\rangle$, is the composition $A\wedge B\xrightarrow{\alpha\wedge\beta}X\wedge X\xrightarrow{\bar\gamma}X$, where $\bar\gamma$ is the reduced commutator map of X. The commutator map and Samelson products are, of course, closely related. In particular, we have:

Proposition 2.2. *Let X be a group-like space such that $X=X_1\times\cdots\times X_l$ as spaces and let $\epsilon_k:X_k\to X$ and $\pi_k:X\to X_k$ denote the inclusion and the projection respectively. Then* nil$X<k$ *if*

$$\pm(\epsilon_{i_1}\circ\pi_{i_1})\circ\langle\pm\epsilon_{j_1},\pm(\epsilon_{i_2}\circ\pi_{i_2})\circ\langle\cdots\pm(\epsilon_{i_k}\circ\pi_{i_k})\circ\langle\pm\epsilon_{j_k},\pm\epsilon_{j_{k+1}}\rangle\cdots\rangle\rangle=0$$

for each $1\le i_1,\dots,i_k,j_1,\dots,j_{k+1}\le l$.

Proof. Consider the group $[X^{k+1},X]$ on which the group structure is given by the pointwise multiplication. Denote the commutator in the group $[X^{k+1},X]$ by $[-,-]$. Then, by definition, the commutator $[\lambda_1,[\cdots[\lambda_k,\lambda_{k+1}]\cdots]]$ of $\lambda_1,\dots,\lambda_{k+1}\in[X^{k+1},X]$ is the composition:

$$X^{k+1}\xrightarrow{\Delta}X^{(k+1)^2}\xrightarrow{\lambda_1\times\cdots\times\lambda_{k+1}}X^{k+1}\xrightarrow{\gamma_k}X,$$

where $\Delta:X^{k+1}\to X^{(k+1)^2}$ is the diagonal map and $\gamma_k:X^{k+1}\to X$ is the k-iterated commutator map as in the previous section. Let $\rho_i:X^{k+1}\to X$ denote the i-th projection. Then it follows from

$$(\rho_1\times\cdots\times\rho_{k+1})\circ\Delta=1_{X^{k+1}} \tag{3}$$

that

$$\gamma_k=[\rho_1,[\cdots[\rho_k,\rho_{k+1}]\cdots]]. \tag{4}$$

Put $\iota = (\epsilon_1 \circ \pi_1) \cdots (\epsilon_n \circ \pi_n)$, where the right hand side is given by the pointwise multiplication. Then ι is a self-homotopy equivalence of X. We also put $\theta_i = \epsilon_i \circ \pi_i$. Let us concentrate in the subgroup $\mathcal{K}$ of $[X^{k+1}, X]$ generated by $\theta_i \circ \rho_j \circ \iota^{-1}$ for $1 \le i \le l$ and $1 \le j \le k+1$. Then, for (4), we have $\gamma_k \in \mathcal{K}$ and hence, by applying Lemma 2.1 to the group $\mathcal{K}$, we obtain that $\gamma_k = 0$ if and only if

$$[\pm(\theta_{i_1} \circ \rho_{j_1} \circ \iota^{-1}), [\cdots [\pm(\theta_{i_k} \circ \rho_{j_k} \circ \iota^{-1}), \pm(\theta_{i_{k+1}} \circ \rho_{j_{k+1}} \circ \iota^{-1})] \cdots]] = 0 \tag{5}$$

for each $1 \le i_1, \ldots, i_{k+1} \le l$ and $1 \le j_1, \ldots, j_{k+1} \le k+1$. Since $\Delta \circ \iota^{-1} = (\iota^{-1} \times \cdots \times \iota^{-1}) \circ \Delta$ and ι is a homotopy equivalence, (5) is equivalent to

$$[\pm(\theta_{i_1} \circ \rho_{j_1}), [\cdots [\pm(\theta_{i_k} \circ \rho_{j_k}), \pm(\theta_{i_{k+1}} \circ \rho_{j_{k+1}})] \cdots]] = 0.$$

Thus, for (3), $\gamma_k = 0$ if

$$\gamma_k \circ (\pm\theta_{i_1} \times \cdots \times \pm\theta_{i_{k+1}}) = 0$$

for each $1 \le i_1, \ldots, i_{k+1} \le l$. Since X is a group-like space, the induced map $[\wedge^{k+1} X, X] \to [X^{k+1}, X]$ from the pinching map $X^{k+1} \to \wedge^{k+1} X$ is monic (see [30, Lemma 1.3.5]). Then it follows that $\gamma_k = 0$ if and only if

$$\langle \pm\theta_{i_1}, \langle \cdots \langle \pm\theta_{i_k}, \pm\theta_{i_{k+1}} \rangle \cdots \rangle\rangle = 0 \tag{6}$$

for each $1 \le i_1, \ldots, i_{k+1} \le l$.

By definition, for maps $f_1 : A_1 \to A_2, f_2 : A_2 \to X, g_1 : B_1 \to B_2, g_2 : B_2 \to X$, we have

$$\langle f_2 \circ f_1, g_2 \circ g_1 \rangle = \langle f_2, g_2 \rangle \circ (f_1 \wedge g_1).$$

Then (6) implies that $\gamma_k = 0$ if and only if

$$\pm\theta_{j_1} \circ \langle \pm\theta_{i_1}, \pm\theta_{j_2} \circ \langle \cdots \pm \theta_{j_k} \circ \langle \pm\theta_{i_k}, \pm\theta_{i_{k+1}} \rangle \cdots \rangle\rangle = 0 \tag{7}$$

for all $1 \le i_1, \ldots, i_{k+1}, j_1, \ldots, j_k \le l$. Moreover, since $(\pi_{i_1} \wedge \cdots \wedge \pi_{i_{k+1}})^* : [X_{i_1} \wedge \cdots \wedge X_{i_{k+1}}, X] \to [\wedge^{k+1} X, X]$ is injective, (7) holds if and only if

$$\pm\theta_{j_1} \circ \langle \pm\epsilon_{i_1}, \pm\theta_{j_2} \circ \langle \cdots \pm \theta_{j_k} \circ \langle \pm\epsilon_{i_k}, \pm\epsilon_{i_{k+1}} \rangle \cdots \rangle\rangle = 0$$

for all $1 \le i_1, \ldots, i_{k+1}, j_1, \ldots, j_k \le l$. Therefore the proof is completed. □

Let p be an odd prime. Before applying Proposition 2.2 to our case, let us recall from [28] some basic facts on the p-primary component of the homotopy groups of spheres.

Proposition 2.3 ([28, Chapter XIII]). *Denote the p-primary component of a finitely generated abelian group A by ${}^pA$. Then we have the following.*

(1) ${}^p\pi_{2n-1+i}(S^{2n-1}) \cong \begin{cases} \mathbf{Z}/p & i = 2p-3 \\ 0 & i \le 4p-7, i \ne 2p-3. \end{cases}$

(2) *Let $\alpha_1(3)$ denote a generator of ${}^p\pi_{2p}(S^3)$ and let $\alpha_1(n) = \Sigma^{n-3}\alpha_1(3)$. Then $\alpha_1(2n-1)$ is a generator of ${}^p\pi_{2n+2p-4}(S^{2n-1})$.*

(3) $\alpha_1(3) \circ \alpha_1(2p) \ne 0$ *and* $\Sigma^2(\alpha_1(3) \circ \alpha_1(2p)) = 0$.

Let X be a simply connected p-complete group-like space of type $(n_1, \ldots, n_l)$. Put $p > n_l - \frac{n_1}{2} + 1$. As is noted above, Kumpel [17] showed that X is p regular if $p \ge n_l - n_1 + 2$ and then we can assume $X = (S^{2n_1-1})^\wedge_p \times \cdots \times (S^{2n_l-1})^\wedge_p$. Denote the inclusion $(S^{2n_i-1})^\wedge_p \to X$ and the projection $X \to (S^{2n_i-1})^\wedge_p$ by ϵ_i and π_i respectively. Then, by Proposition 2.3, we have

$$\pi_s \circ \langle \epsilon_i, \epsilon_j \rangle = \begin{cases} a\alpha_1(2n_s - 1) & \text{if } n_i + n_j = n_s + p - 1 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

here $a \in \mathbf{Z}/p$ is possibly 0. Hence it follows also from Proposition 2.3 that $\pi_s \circ \langle \epsilon_i, (\epsilon_t \circ \pi_t) \circ \langle \epsilon_j, \epsilon_k \rangle\rangle \ne 0$ if and only if

$$\begin{cases} n_s = 2,\ n_i + n_t = p+1,\ n_j + n_k = n_t + p - 1 \\ \pi_2 \circ \langle \epsilon_i, \epsilon_t \rangle \ne 0,\ \pi_t \circ \langle \epsilon_j, \epsilon_k \rangle \ne 0. \end{cases} \tag{9}$$

Therefore, by Lemma 2.2, an easy inspection shows:

Theorem 2.4. *Let X be a simply connected p-complete group-like space of type $(n_1, \ldots, n_l)$. Then we have:*

(1) $\mathrm{nil} X = 1$ *if* $p > 2n_l$.

(2) $\mathrm{nil} X = 1$ *or* 2 *if:*

(a) $\frac{3}{2}n_l < p < 2n_l$.
(b) $n_l - \frac{n_1}{2} + 1 < p \le 2n_l$ *and* X *does not satisfy* (9).

(3) $\mathrm{nil}X = 3$ *if* $n_1 = 2, n_l < p \le \frac{3}{2}n_l$ *and* X *satisfy* (9).

The classical result of James and Thomas [14] yields if the above group-like space X is a loop space and $n_l - n_1 + 2 < p < 2n_l$ then X is not homotopy commutative, that is, $\mathrm{nil}X > 1$. Thus we obtain:

Corollary 2.5. *Let* X *be a simply connected* p*-compact group of type* $(n_1, \dots, n_l)$ *with* $n_1 \le \cdots \le n_l$. *Then we have:*

(1) $\mathrm{nil}X = 1$ *if* $p > 2n_l$.
(2) $\mathrm{nil}X = 2$ *if:*
  (a) $\frac{3}{2}n_l < p < 2n_l$.
  (b) $n_l - \frac{n_1}{2} + 1 < p \le 2n_l$ *and* X *does not satisfy* (9).
(3) $\mathrm{nil}X = 3$ *if* $n_1 = 2, n_l < p \le \frac{3}{2}n_l$ *and* X *satisfy* (9).

In most cases, the above corollary reduces the proof of Theorem 1.6 and Theorem 1.7 to examining (9) case by case when $n_l < p < \frac{3}{2}n_l$.

Hereafter we will use the following notation. Let X be a p-compact group such that

$$X \simeq (S^{2n_1-1})^\wedge_p \times \cdots \times (S^{2n_l-1})^\wedge_p.$$

Then we denote the inclusion $(S^{2n_i-1})^\wedge_p \to X$ and the projection $X \to (S^{2n_i-1})^\wedge_p$ by ϵ_i and π_i respectively.

3. Lie groups

The latter part of Theorem 1.6 immediately follows from a dimensional reason of Corollary 2.5 (see [22] for the types of simple Lie groups). In the case $(G, p) = (G_2, 7), (F_4, 13), (E_6, 13), (E_7, 19), (E_8, 31)$, Hamanaka and Kono [9] showed that $\pi_2 \circ \langle \epsilon_2, \epsilon_n \rangle \ne 0$, where n is the largest entry in the type of G, and then, for Corollary 2.5, we obtain $\mathrm{nil}G^\wedge_p = 3$. Thus the remaining cases of (G, p) are listed in the following table and we will check (9) in these cases, where $[x]$ in the table stands for the largest integer less than or equal to x.

G	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	$\mathrm{Spin}(n)$	E_7	E_8
p	$n \le p \le \frac{3}{2}n$	$2n < p < 3n$	$2[\frac{n}{2}] < p < 3[\frac{n}{2}]$	23	37

3.1. **Classical groups.**

3.1.1. $\mathrm{SU}(n)$. Recall that the type of $\mathrm{SU}(n)$ is $(2, 3, \dots, n)$. Let p be a prime such that $p \ge n$. The classical result of Bott [3] shows that if $i + j > n$, the order of the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is a non-zero multiple of

$$\nu_p\left(\frac{(i+j-1)!}{(i-1)!(j-1)!}\right), \tag{10}$$

where $\nu_p(p^k q) = p^k$ for $(p, q) = 1$.

It follows from (10) that $\langle 1_{\mathrm{SU}(2)^\wedge_2}, 1_{\mathrm{SU}(2)^\wedge_2} \rangle \ne 0$ and then $\mathrm{nilSU}(2)^\wedge_2 \ge 2$. Since $\mathrm{SU}(2) \cong S^3$, $\pi_9(SU(2)) \cong \mathbf{Z}/9$ (see [28]). Then $\langle 1_{\mathrm{SU}(2)^\wedge_2}, \langle 1_{\mathrm{SU}(2)^\wedge_2}, 1_{\mathrm{SU}(2)^\wedge_2} \rangle\rangle = 0$ and therefore $\mathrm{nilSU}(2)^\wedge_2 = 2$.

Put $2 < n < p \le \frac{3}{2}n$. Then it also follows from (2.3) and (10) that $\pi_2 \circ \langle \epsilon_n, \epsilon_{p-n+1} \rangle \ne 0$ and $\pi_{p-n+1} \circ \langle \epsilon_n, \epsilon_{2p-2n} \rangle \ne 0$, and hence, for (9), $\pi_2 \circ \langle \epsilon_n, (\epsilon_{p-n+1} \circ \pi_{p-n+1}) \circ \langle \epsilon_n, \epsilon_{2p-2n} \rangle\rangle \ne 0$. Therefore, by Lemma 2.2 and Corollary 2.5, we have obtained $\mathrm{nilSU}(n)_{(p)} = 3$.

Put $n = p$. This is the only one case which is not covered by Corollary 2.5. By an analogous calculation to the above case, one has $\pi_2 \circ \langle \epsilon_{p-1}, (\epsilon_2 \circ \pi_2) \circ \langle \epsilon_{p-1}, \epsilon_2 \rangle\rangle \ne 0$ and then $\mathrm{nilSU}(p)_{(p)} \ge 3$. Recall from [28] that $\Sigma^2 : {}^p\pi_{2n+2k}(S^{2n-1}) \to {}^p\pi_{2n+2k+2}(S^{2n+1})$ is the zero map. Then $\pi_s \circ \langle \pm\epsilon_i, \pm(\epsilon_t \circ \pi_t) \circ \langle \pm\epsilon_j, \pm(\epsilon_u \circ \pi_u) \circ \langle \pm\epsilon_k, \pm\epsilon_l \rangle\rangle\rangle = 0$ for each $1 \le i, j, k, l, s, t, u \le p$ and hence it follows from Lemma 2.2 that $\mathrm{nilSU}(p)_{(p)} = 3$.

3.1.2. $\mathrm{Sp}(n)$. Recall that the type of $\mathrm{Sp}(n)$ is $(2, 4, \dots, 2n)$. Let $p > 2n$. The result of Bott [3] also shows that if $i + j > n$, the order of the Samelson product $\langle \epsilon_{2i}, \epsilon_{2j} \rangle$ is a non-zero multiple of

$$\nu_p\left(\frac{(2i+2j-1)!}{(2i-1)!(2j-1)!}\right).$$

Then, by an analogous calculation to the case of $\mathrm{SU}(n)$, we can deduce $\mathrm{nilSp}(n)^\wedge_p = 3$ if $2n < p < 3n$.

3.1.3. Spin(n). Friedlander [8] showed that if p is an odd prime, there exists an equivalence of loop spaces

$$\mathrm{Spin}(2n+1)^{\wedge}_p \simeq \mathrm{Sp}(n)^{\wedge}_p.$$

Then it follows from the above result on Sp(n) that $\mathrm{nilSpin}(2n+1)_{(p)} = 3$ if $2n < p < 3n$. On the other hand, Harris [10] showed that the canonical fiber sequence

$$\mathrm{Spin}(2n) \to \mathrm{Spin}(2n+1) \to S^{2n+1}$$

splits after the completion at an odd prime and thus, by Corollary 2.5, $\mathrm{nilSpin}(2n)^{\wedge}_p = 3$ if $2n < p < 3n$.

3.2. **Exceptional groups.** We first show a way to find a non-trivial Samelson product by using the Steenrod operations which is used in [16] and also in [9]. Let X be a p-regular connected p-compact group of type $(n_1, \dots, n_l)$. Then we can set

$$H^*(BX; \mathbf{F}_p) = \mathbf{F}_p[x_1, \dots, x_l],\ |x_i| = 2n_i,\ \epsilon_i^*(\sigma(x_i)) = u_{2n_i - 1}, \tag{11}$$

where σ is the cohomology suspension and u_n is a generator of $H^n(S^n; \mathbf{Z}/p)$.

Lemma 3.1. *Let θ be a mod p Steenrod operation. If θx_i contains the term $a x_j x_k$ with $a \neq 0 \in \mathbf{F}_p$, then $\langle \epsilon_j, \epsilon_k \rangle \neq 0$.*

Proof. Suppose that $\langle \epsilon_j, \epsilon_k \rangle = 0$. Let $\mathrm{ad} : [V, \Omega W] \to [\Sigma V, W]$ denote the adjoint congruence. Then the Whitehead product $[\mathrm{ad}\epsilon_j, \mathrm{ad}\epsilon_k] = \mathrm{ad}\langle \epsilon_j, \epsilon_k \rangle = 0$ and hence there exists a map $f : (S^{2n_j})^{\wedge}_p \times (S^{2n_k})^{\wedge}_p \to BX$ such that the following square diagram is homotopy commutative.

$$\begin{array}{ccc} (S^{2n_j})^{\wedge}_p \vee (S^{2n_k})^{\wedge}_p & \xrightarrow{\mathrm{ad}\epsilon_j \vee \mathrm{ad}\epsilon_k} & BX \vee BX \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \nabla} \\ (S^{2n_j})^{\wedge}_p \times (S^{2n_k})^{\wedge}_p & \xrightarrow{f} & BX, \end{array}$$

where ι and ∇ are the inclusion and the folding map respectively. It follows from (11) that

$$(\mathrm{ad}\epsilon_m)^*(x_m) = u_{2n_m}$$

for $m = 1, \dots, l$. Then if θx_i contains the terms $a x_j x_k$ with $a \neq 0 \in \mathbf{F}_p$,

$$f^*(\theta x_i) = a u_{2n_j} \times u_{2n_k} \neq 0.$$

On the other hand,

$$f^*(\theta x_i) = \theta f^*(x_i) = 0$$

and this is a contradiction. Thus Lemma 3.2 is established. ☐

Corollary 3.2. *Let $n_l < p \leq \frac{3}{2} n_l$ and let $n_1 < \cdots < n_l$. Suppose that $n_1 = 2$, $\mathcal{P}^1 x_1$ and $\mathcal{P}^1 x_l$ contain the terms $a x_i x_l$ and $b x_j x_k$ for $a, b \neq 0 \in \mathbf{F}_p$ respectively. Then* $\mathrm{nil} X = 3$.

Proof. For $n_1 < \ldots < n_l$ and Proposition 2.3, one has $\pi_n \circ \langle \epsilon_i, \epsilon_l \rangle = 0$ unless $n = 1$. On the other hand, it follows from Lemma 3.1 that $\langle \epsilon_i, \epsilon_l \rangle \neq 0$ and then $\pi_1 \circ \langle \epsilon_i, \epsilon_l \rangle \neq 0$. Analogously, one can see that $\pi_l \circ \langle \epsilon_j, \epsilon_k \rangle \neq 0$ and then, for (9) and Lemma 2.5, we have established Corollary 3.2. ☐

We next prepare notation for some symmetric polynomials. Denote c_k the k-th elementary symmetric function in $t_1, \dots, t_n$ for $k = 1, \dots, n$, that is,

$$1 + c_1 + \cdots + c_n = \prod_{i=1}^{n} (1 + t_i).$$

Define symmetric polynomials p_k and s_k for $k = 1, \dots, n$ by

$$1 - p_1 + \cdots + (-1)^n p_n = \prod_{i=1}^{n} (1 - t_i^2)$$

and

$$s_k = t_1^{2k} + \cdots + t_n^{2k}$$

respectively. Then one has Girard's formula

$$s_k = (-1)^k k \sum_{i_1 + 2i_2 + \cdots + n i_n = k} (-1)^{i_1 + \cdots + i_n} \frac{(i_1 + \cdots + i_n - 1)!}{i_1! \cdots i_n!} p_1^{i_1} \cdots p_n^{i_n} \tag{12}$$

(see [21]). In the canonical way, we will identify c_k and p_k with the universal k-th Chern class and Pontrjagin class respectively.

3.2.1. *E_7 with $p = 23$.* Recall that we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}(10) & = & \mathrm{Spin}(10) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ E_6 & \xrightarrow{j} & E_7 \end{array} \tag{13}$$

Recall also that the mod 23 cohomology of $B\mathrm{Spin}(10)$, BE_6 and BE_7 are:

$$H^*(B\mathrm{Spin}(10); \mathbf{F}_{23}) = \mathbf{F}_{23}[p_1, p_2, p_3, p_4, c_5],\ |p_i| = 4i, |c_5| = 10,$$
$$H^*(BE_6; \mathbf{F}_{23}) = \mathbf{F}_{23}[x_2, x_5, x_6, x_8, x_9, x_{12}],\ |x_i| = 2i,$$
$$H^*(BE_7; \mathbf{F}_{23}) = \mathbf{F}_{23}[y_2, y_6, y_8, y_{10}, y_{12}, y_{14}, y_{18}],\ |y_i| = 2i.$$

In [9], it is shown that x_i and y_i can be chosen such that:

$$j^*(y_i) = x_i\ (i = 2, 6, 8), \quad j^*(y_{10}) = x_5^2, \quad j^*(y_{14}) = x_5x_9,$$
$$i_1^*(x_2) = p_1, \quad i_1^*(x_5) = c_5, \quad i_1^*(x_6) = -6p_3 + p_1p_2,$$
$$i_1^*(x_8) = 12p_4 + p_2^2 - \tfrac{1}{2}p_1^2p_2.$$

Then, for (13), one has:

$$\begin{aligned} i_2^*(y_2) &= p_1, & i_2^*(y_6) &= -6p_3 + p_1p_2, \\ i_2^*(y_8) &= 12p_4 + p_2^2 - \tfrac{1}{2}p_1^2p_2, & i_2^*(y_{10}) &= c_5^4. \end{aligned} \tag{14}$$

For a dimensional reason, one also has $i_1^*(x_9) = ap_1^2c_5 - bp_2c_5$ for $a, b \in \mathbf{F}_{23}$ and then it follows from (13) and (14) that

$$i_2^*(y_{14}) = ap_1^2c_5^2 - bp_2c_5^2. \tag{15}$$

We can see from the following proposition that $(E_7)^\wedge_{23}$ satisfies the condition of Corollary 3.2 and then $\mathrm{nil}(E_7)^\wedge_{23} = 3$.

Proposition 3.3. *$\mathcal{P}^1y_2$ contains the term cy_6y_{18} with $c \neq 0$ and $\mathcal{P}^1y_6$ contains the term $dy_{10}y_{18} + ey_{14}y_{14}$ with $d \neq 0$ or $e \neq 0$.*

Proof. Define a ring homomorphism

$$\pi : \mathbf{F}_{23}[p_1, \dots, p_4, c_5] \to \mathbf{F}_{23}[a_2, \dots, a_4, b_5]/(a_2^3, a_3^2, a_4^2, b_5^3, 12a_4 + a_2^2)$$

by

$$\pi(p_1) = 0,\ \pi(p_i) = a_i\ (i = 2, 3, 4),\ \pi(c_5) = b_5.$$

Then, for (14) and (15), one has

$$\pi(i_2^*(y_2)) = \pi(i_2^*(y_8)) = \pi(i_2^*(y_6^2)) = \pi(i_2^*(y_{14}y_{10})) = 0. \tag{16}$$

Put $\mathcal{P}^1y_2 = cy_6y_{18}$+other terms for $c \in \mathbf{F}_{23}$. Then, for (16) and a degree reason, one has

$$\pi(i_2^*(\mathcal{P}^1y_2)) = c\pi(i_2^*(y_6y_{18})).$$

On the other hand, since $p_1 = s_1$ and $\mathcal{P}^1s_1 = 2s_{12}$, Girard's formula (12) yields that

$$\pi(i_2^*(\mathcal{P}^1y_2)) = \pi(\mathcal{P}^1i_2^*(y_2)) = \pi(\mathcal{P}^1p_1) = -15a_3a_4b_5^2 \neq 0.$$

Thus we have $c \neq 0$.

We define a ring homomorphism

$$\pi' : \mathbf{F}_{23}[p_1, \dots, p_4, c_5] \to \mathbf{F}_{23}[a_2', a_4', b_5']/((a_2')^3, (a_4')^2, (b_5')^5, 12a_4' + (a_2')^2)$$

by

$$\pi'(p_i) = 0\ (i = 1, 3), \pi'(p_j) = a_j'\ (j = 2, 4),\ \pi'(c_5) = b_5'.$$

Then, for (14) and (15), we have

$$\pi'(i_2^*(y_2)) = \pi'(i_2^*(y_6)) = \pi'(i_2^*(y_8)) = 0. \tag{17}$$

Put $\mathcal{P}^1y_6 = dy_{10}y_{18} + ey_{14}y_{14}$+other terms for $d, e \in \mathbf{F}_{23}$. Then it follows from (17) and a degree reason that

$$\pi'(i_2^*(\mathcal{P}^1y_6)) = d\pi'(i_2^*(y_{10}y_{18})) + e\pi'(i_2^*(y_{14}y_{14})).$$

For Girard's formula (12), we have:

$$\pi'(\mathcal{P}^1 p_1) = \pi'(\mathcal{P}^1 s_1) = \pi'(2s_{12}) = -a_2'(b_5')^4$$
$$\pi'(\mathcal{P}^1 s_3) = \pi'(6s_{14}) = -9a_4'(b_5')^4$$

Since $s_3 = p_1^3 - 3p_1p_2 + 3p_3$, one has $\pi(\mathcal{P}^1 p_3) = 9a_4'(b_5')^4$ and then, for (14),

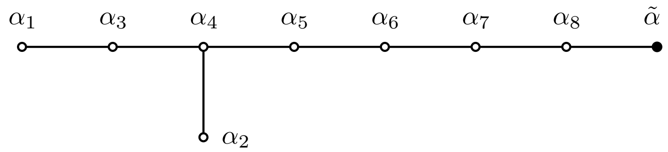
$$\pi'(\mathcal{P}^1(i_2^*(y_6))) = \pi'(\mathcal{P}^1(-6p_3 + p_1p_2)) = -19a_4'(b_5')^4 \neq 0.$$

Therefore we have obtained $d \neq 0$ or $e \neq 0$ and the proof is completed. □

3.2.2. *E_8 with $p = 37$.* Recall that the mod 37 cohomology of BE_8 is given by

$$H^*(BE_8; \mathbf{F}_{37}) = \mathbf{F}_{37}[z_2, z_8, z_{12}, z_{14}, z_{18}, z_{20}, z_{24}, z_{30}],\ |z_i| = 2i.$$

In order to see the action of $\mathcal{P}^1$ on $H^*(BE_8; \mathbf{F}_{37})$, we shall choose suitable z_i. Let α_i $(i = 1, \dots, 8)$ and $\tilde{\alpha}$ be respectively the simple roots and the dominant root of E_8 as indicated in the following extended Dynkin diagram of E_8 (see [22] for details).



Denote the Weyl group of E_8 by W. We consider the subgroup K of W which is generated by the reflections corresponding to α_i for $i = 2, \dots, 8$ and $\tilde{\alpha}$. Then, by choosing appropriate generators $t_1, \dots, t_8 \in H^2(BT; \mathbf{F}_{23})$, one has

$$H^*(BT; \mathbf{F}_{37})^K = \mathbf{F}_{37}[p_1, \dots, p_7, c_8],$$

where T is a maximal torus of E_8. This is nothing but the cohomology of $B\mathrm{Spin}(16)$ in BE_8. Let φ be the elements of W corresponding to α_1. Then, by definition, W is generated by K and φ. Hence one has

$$H^*(BT; \mathbf{F}_{37})^W = H^*(BT; \mathbf{F}_{37})^\varphi \cap \mathbf{F}_{37}[p_1, \dots, p_7, c_8].$$

On the other hand, the canonical map $i : BT \to BE_8$ induces an isomorphism

$$i^* : H^*(BE_8; \mathbf{F}_{37}) \xrightarrow{\cong} H^*(BT; \mathbf{F}_{37})^W.$$

It is shown in [9] that

$$\varphi(c_1) = -c_1,\ \varphi(c_2) = c_2,\ \varphi(c_8) = c_8 - \tfrac{1}{4}c_1c_7,\ \varphi(p_1) = p_1$$

and

$$\varphi(p_i) \equiv p_i + c_1h_i \mod (c_1^2) \tag{18}$$

for $i = 2, \dots, 7$, where

$$\begin{aligned} h_2 &= \tfrac{3}{2}c_3, & h_3 &= -\tfrac{1}{2}(5c_5 + c_2c_3), \\ h_4 &= \tfrac{1}{2}(7c_7 + 3c_2c_5 - c_3c_4), & h_5 &= -\tfrac{1}{2}(5c_2c_7 - 3c_3c_6 + c_4c_5), \\ h_6 &= -\tfrac{1}{2}(5c_3c_8 - 3c_4c_7 + c_5c_6), & h_7 &= \tfrac{1}{2}(3c_5c_8 - c_6c_7). \end{aligned}$$

Then it is immediate that

$$i^*(z_2) = p_1. \tag{19}$$

By a direct calculation, Hamanaka and Kono [9] showed:

Proposition 3.4 (Hamanaka and Kono [9])**.** *If $d_8 \in H^{16}(BT; \mathbf{F}_{37})$ and $d_{12} \in H^{24}(BT; \mathbf{F}_{37})$ satisfy $\varphi(d_8) \equiv d_8 \mod (c_1^2)$ and $\varphi(d_{12}) \equiv d_{12} \mod (c_1^2, c_2^2)$, then*

$$d_8 \equiv a\tilde{z}_8 \mod (p_1^4),\ d_{12} \equiv b\tilde{z}_{12} \mod (p_1^2)$$

for $a, b \in \mathbf{F}_{37}$ and

$$\tilde{z}_8 = 120p_4 + 1680c_8 + p_1^2p_2 - 36p_1p_3 + 10p_2^2,$$
$$\tilde{z}_{12} = 60p_6 - p_1p_2p_3 - 5p_1p_5 + \tfrac{5}{36}p_2^3 - 5p_2p_4 + 110p_2c_8 + 3p_3^2.$$

Since $\tilde{z}_2, \tilde{z}_8, \tilde{z}_{12}$ are algebraically independent, we have:

Corollary 3.5 (Hamanaka and Kono [9]). *Let $\tilde{z}_8$ and $\tilde{z}_{12}$ be as in Proposition 3.4. Then we can choose generators z_8 and z_{12} of $H^*(BE_8;\mathbf{F}_{37})$ such that*

$$i^*(z_8)\equiv\tilde{z}_8,\ i^*(z_{12})\equiv\tilde{z}_{12}\mod (p_1^2).$$

Let us consider the generator z_{14}. For a dimensional reason, an element of degree 28 in $\mathbf{F}_{37}[p_1,\dots,p_7,c_8]$ is given by a linear combination

$$ap_7+bp_2^2p_3+cp_2p_5+dp_3c_8+ep_3p_4\mod (p_1)$$

for $a,b,c,d,e\in\mathbf{F}_{37}$. It is straightforward to check that

$$\begin{aligned}\varphi(p_2^2p_3)&\equiv p_2^2p_3+6c_1c_3^3c_4-12c_1c_3c_4c_6-10c_1c_4^2c_5,\\ \varphi(p_2p_5)&\equiv p_2p_5+3c_1c_3^2c_7+\tfrac{3}{2}c_1c_3c_5^2-c_1c_4^2c_5,\\ \varphi(p_3c_8)&\equiv p_3c_8-\tfrac{1}{4}c_1c_3^2c_7-\tfrac{5}{2}c_1c_5c_8+\tfrac{1}{2}c_1c_6c_7,\\ \varphi(p_3p_4)&\equiv p_3p_4-\tfrac{1}{2}c_1c_3^3c_4+\tfrac{7}{2}c_1c_3^2c_7+c_1c_3c_4c_6\\ &\quad+5c_1c_3c_5^2-\tfrac{5}{2}c_1c_4^2c_5-5c_1c_5c_8-7c_1c_6c_7\mod (c_1^2,c_2).\end{aligned}$$

Then it follows that:

Proposition 3.6. *If $d_{14}\in H^{28}(BT;\mathbf{F}_{37})$ satisfy $\varphi(d_{14})\equiv d_{14}\mod (c_1^2,c_2)$, then*

$$d_{14}\equiv a\tilde{z}_{14}\mod (p_1)$$

for $a\in\mathbf{F}_{37}$ and

$$\tilde{z}_{14}=480p_7-p_2^2p_3+40p_2p_5-12p_3p_4+312p_3c_8.$$

Since $\tilde{z}_2,\tilde{z}_8,\tilde{z}_{12},\tilde{z}_{14}$ are algebraically independent, we obtain:

Corollary 3.7. *Let $\tilde{z}_{14}$ be as in Proposition 3.6. We can choose a generator z_{14} of $H^*(BE_8;\mathbf{F}_{37})$ such that*

$$i^*(z_{14})\equiv\tilde{z}_{14}\mod (p_1).$$

We choose generators z_2,z_8,z_{12},z_{14} of $H^*(BE_8;\mathbf{F}_{37})$ as in (19), Corollary 3.5 and Corollary 3.7. For the following proposition, we can see that $(E_8)^\wedge_{37}$ satisfies the condition of Corollary 3.2 and then $\mathrm{nil}(E_8)^\wedge_{37}=3$.

Proposition 3.8. *$\mathcal{P}^1z_2$ and $\mathcal{P}^1z_8$ contain the terms az_8z_{30} and $bz_{20}z_{24}$ with $a,b\neq 0$ respectively.*

Proof. Consider the ring homomorphism

$$\pi:\mathbf{F}_{37}[p_1,\dots,p_7,c_8]\to\mathbf{F}_{37}[a_3,a_4,a_7,b_8]/(a_3^2,a_4^2,a_7^2,b_8^4,a_3a_4-26a_3b_8-40a_7)$$

defined by

$$\pi(p_i)=0\ (i=1,2,5,6),\ \pi(p_j)=a_j\ (j=3,4,7),\ \pi(c_8)=b_8.$$

Then, for (19), Corollary (3.5) and Corollary 3.7, we have

$$\pi(i^*(z_2))=\pi(i^*(z_{12}))=\pi(i^*(z_{14}))=0$$

and, for a degree reason, we also have

$$\pi(i^*(z_{18}))=0.$$

Put $\mathcal{P}^1z_2=az_8z_{30}$+other terms. Thus one has

$$\pi(i^*(\mathcal{P}^1z_2))=a\pi(i^*(z_8z_{30})).$$

On the other hand, it follows from Girard's formula (12) and (19) that

$$\pi(i^*(\mathcal{P}^1z_2))=\pi(\mathcal{P}^1i^*(z_2))=\pi(\mathcal{P}^1p_1)=\pi(\mathcal{P}^1s_1)=\pi(2s_{19})=2a_4a_7b_8^2\neq 0$$

and then $a\neq 0$.

Define a ring homomorphism

$$\pi':\mathbf{F}_{37}[p_1,\dots,p_7,c_8]\to\mathbf{F}_{37}[a_2',a_4',b_8']/((a_2')^2,(a_4')^6,(b_8')^6,a_4+14b_8)$$

by

$$\pi'(p_i)=0\ (i=1,3,5,6,7),\ \pi'(p_j)=a_j'\ (j=2,4),\ \pi'(c_8)=b_8'.$$

Then, for (19), Corollary 3.5 and Corollary (3.7), we have

$$\pi'(i^*(z_4))=\pi'(i^*(z_8))=\pi'(i^*(z_{14}))=\pi'(i^*(z_{12}^2))=0.$$

Put $\mathcal{P}^1 z_8 = b y_{20} z_{24}$+other terms. Then we can see that

$$\pi'(i^*(\mathcal{P}^1 z_8)) = b\pi'(i^*(z_{20} z_{24})).$$

Let us make a direct calculation of $\pi'(i^*(\mathcal{P}^1 z_8))$. It follows from Girard's formula (12) that:

$$\pi'(\mathcal{P}^1 s_2) = \pi'(4s_{20}) = -6(b_8)^5 \qquad \pi'(\mathcal{P}^1 s_4) = \pi'(8s_{22}) = 16a_2'(b_8')^5$$
$$\pi'(\mathcal{P}^1 c_8) = \pi'(s_{18} c_8) = 26a_2'(b_8')^5$$

Since $s_2 = p_1^2 - 2p_2$ and $s_4 = p_1^4 - 4p_1^2 p_2 + 4p_1 p_3 + 2p_2^2 - 4p_4$, one has

$$\pi'(\mathcal{P}^1 p_2) = 3(b_8')^5,\ \pi'(\mathcal{P}^1 p_4) = -a_2'(b_8')^5$$

and then, for Corollary 3.5,

$$\begin{aligned}\pi'(i^*(\mathcal{P}^1 z_8)) &= \pi'(\mathcal{P}^1 i^*(z_8))\\ &= 120\pi'(\mathcal{P}^1 p_4) + 1680\pi'(\mathcal{P}^1 c_8) + 20a_2'\pi'(\mathcal{P}^1 p_2)\\ &= -3a_2'(b_8')^5 \neq 0.\end{aligned}$$

Hence we have obtained $b \neq 0$ and thus the proof is completed. □

4. Exotic p-compact groups

By consulting Clark and Ewing's list [5], we can easily see that Theorem 1.7 follows from a dimensional reason of Corollary 2.5 except for exotic p-compact groups X of type $(n_1, \dots, n_l)$ satisfying

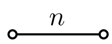
$$(N(X), p) = (2\text{b}, n_l + 1), (23, 11), (30, 31).$$

Let K be the Weyl group of X. Then the mod p cohomology of X is the invariant ring of K, $\mathbf{F}_p[t_1, \dots, t_l]^K = \mathbf{F}_p[x_1, \dots, x_l]$, in which $|t_i| = 2$ and $x_i = 2n_i$. In particular, we have $|x_1| = 4$. Note that we can apply Corollary 3.2 to the p-compact groups corresponding to K and the condition in Corollary 3.2 is equivalent to:

$$\mathcal{P}^1 x_1 = a x_i x_l + \text{other terms},\ \mathcal{P}^1 x_l = b x_j x_k + \text{other terms} \tag{20}$$

for some i, j, k, l and $a, b \neq 0 \in \mathbf{F}_p$, where the action of $\mathcal{P}^1$ comes from the relation $\mathcal{P}^1 t_i = t_i^p$. Then we will show that K satisfies this condition.

4.1. $N(X) = 2$b. The pseudoreflection group of number 2b in Clark and Ewing's list [5] is the Coxeter group $I_2(n)$ for $n \geq 3$ which corresponds to the following Coxeter diagram, where $p \equiv \pm 1\ (n)$ (see [13]).



For $p \equiv 1\ (n)$, it is the dihedral group of order $2n$, denoted $\mathcal{D}_{2n}$, acting on $\mathbf{F}_p^2$ via the matrices:

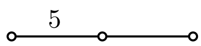
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

where ω is an n-th root of unity which exists in $\mathbf{F}_p$ since $p \equiv 1\ (n)$. Then the invariant ring of $\mathcal{D}_{2n}$ is

$$\mathbf{F}_p[t_1 t_2, t_1^n + t_2^n]. \tag{21}$$

Put $p = n + 1$ and $x_2 = t_1 t_2$, $x_n = t_1^n + t_2^n$. Then we have $\mathcal{P}^1 x_2 = x_2 x_n$ and then, by (20), the proof of Theorem 1.7 for $(N(X), p) = (2\text{b}, n + 1)$ is completed.

4.2. $(N(X), p) = (23, 11)$. The pseudoreflection group of number 23 in Clark and Ewing's list [5] is the Coxeter group H_3 corresponding to the Coxeter diagram:



Let P_1 be the invariant ring of $I_2(5)$ over $\mathbf{F}_{11}$. Then it follows from (21) that $P_1 = \mathbf{F}_{11}[x_2, x_n]$ such that

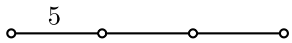
$$\mathcal{P}^1 x_2 = x_2 x_5^2 - 2x_2^6. \tag{22}$$

Denote the invariant ring of H_3 over $\mathbf{F}_{11}$ by P_2. Then we have $P_2 = \mathbf{F}_{11}[y_2, y_6, y_{10}]$, $|y_i| = 2i$ in which we can set $i(y_2) = x_2$ by the map $i : P_2 \to P_1$ induced from the inclusion of the above Coxeter diagrams. Put $\mathcal{P}^1 y_2 = a y_2 y_{10}$+other terms for $a \in \mathbf{F}_{11}$. Then it follows from (22) and a degree reason that

$$x_2 x_5^2 \equiv \mathcal{P}^1 x_2 \equiv \mathcal{P}^1 i(y_2) \equiv i(\mathcal{P}^1 y_2) \equiv a i(y_2) i(y_{10}) \mod (x_2^2)$$

and hence $a \neq 0$. Therefore the invariant ring of H_3 over $\mathbf{F}_{11}$ satisfies (20).

4.3. $(N(X), p) = (30, 31)$**.** The pseudoreflection group of number 30 in Clark and Ewing list [5] is the Coxeter group H_4 corresponding to the Coxeter diagram:



It follows from (21) that the invariant ring of $I_2(5)$ over $\mathbf{F}_{31}$ is given by $Q_1 = \mathbf{F}_{31}[x_2, x_5]$ for $x_2 = t_1 t_2$ and $x_5 = t_1^5 + t_2^5$. Then we have

$$\mathcal{P}^1 x_2 \equiv x_2 x_5^6,\ \mathcal{P}^1 x_5 \equiv 5x_5^7 \mod (x_2^2). \tag{23}$$

Let $Q_2 = \mathbf{F}_{31}[y_2, y_6, y_{10}],\ |y_i| = 2i$ be the invariant ring of H_3 over $\mathbf{F}_{31}$, where we put $i(y_2) = x_2$ by the canonical by $i : Q_2 \to Q_1$ as above. Put $\mathcal{P}^1 y_2 = a y_2 y_{10}^3$+other terms for $a \in \mathbf{F}_{31}$. Then, for (23) and a degree reason, we have

$$x_2 x_5^6 \equiv \mathcal{P}^1 x_2 \equiv \mathcal{P}^1 i(y_2) \equiv i(\mathcal{P}^1 y_2) \equiv a i(y_2) i(y_{10}^3) \mod (x_2^2)$$

and hence $a \neq 0$. Thus we can put $i(y_{10}) = x_5^2$+other terms. Therefore we obtain

$$\mathcal{P}^1 y_2 = y_2 y_{10}^3,\ \mathcal{P}^1 y_{10} = 10 y_{10}^4 \mod (y_2^2, y_6). \tag{24}$$

Denote by Q_3 the invariant ring of H_4 over $\mathbf{F}_{31}$, $\mathbf{F}_{31}[z_2, z_{12}, z_{20}, z_{30}],\ |z_i| = 2i$, in which we can set $j(z_2) = y_2$ by the canonical map $j : Q_3 \to Q_2$. Put $\mathcal{P}^1 z_2 = b z_2 z_{30} + c z_{12} z_{20}$+other terms for $b, c \in \mathbf{F}_{31}$. Then, for (24), we have

$$\begin{aligned} y_2 y_{10}^3 &\equiv \mathcal{P}^1 y_2 \equiv \mathcal{P}^1 j(z_2) \equiv j(\mathcal{P}^1 z_2) \\ &\equiv b j(z_2) j(z_{30}) + c j(z_{12}) j(z_{20}) \mod (y_2^2, y_6). \end{aligned} \tag{25}$$

Hence $b \neq 0$ or $c \neq 0$. Suppose that $b = 0$. Then, for (25) we can set $j(z_{20}) = y_{10}^2$+other terms. Put $\mathcal{P}^1 z_{20} = d z_{20} z_{30}$+other terms. Thus, for (23) and a degree reason, we have

$$20 y_{10}^5 \equiv \mathcal{P}^1 y_{10}^2 \equiv \mathcal{P}^1 j(z_{20}) \equiv j(\mathcal{P}^1 z_{20}) \equiv d j(z_{20}) j(z_{30}) \mod (y_2^2, y_6)$$

and hence $d \neq 0$. Summarizing, we have established that $\mathcal{P}^1 z_2 = b z_2 z_{30} + c z_{12} z_{20}$+other terms with $b \neq 0$ or $c \neq 0$ such that if $b = 0$, $\mathcal{P}^1 z_{20} = d z_{20} z_{30}$+other terms with $d \neq 0$. This satisfies (20) and now the proof of Theorem 1.7 is completed.

References

[1] K.K.S. Andersen, J. Grodal, J.M. Møller and A. Viruel, *The classification of p-compact groups for p odd*, Ann. of Math. **167** (2008), 95-210.

[2] J. Aguadé and L. Smith, *On the mod p torus theorem of John Hubbuck*, Math. Z. **191** (1986), no. 2, 325-326.

[3] R. Bott, *A note on the Samelson products in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.

[4] A.K. Bousfield, D.M. Kan, *Homotopy Limits, Completions and Localizations*, LNM **304** (1972), Springer-Verlag, Berlin-New York.

[5] A. Clark and J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425-434.

[6] W.G. Dwyer and C.W. Wilkerson, *A new finite loop space at the prime two*, J. Amer. Math. Soc. **6** (1993), 37-64.

[7] W.G. Dwyer and C.W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. **139** (1994), 395-442.

[8] E.M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. of Math. **101** (1975), 510-520.

[9] H. Hamanaka and A. Kono, *A note on Samelson products in exceptional Lie groups*, preprint available at `http://www.math.kyoto-u.ac.jp/preprint/2008/15kono.pdf`.

[10] B. Harris, *Suspensions and characteristic maps for symmetric spaces*, Ann. of Math. **76** (1962), 295-305.

[11] Y. Hemmi, *Higher homotopy commutativity of H-spaces and the mod p torus theorem*, Pacific J. Math. **149** (1991), no. 1, 95-111.

[12] J. Hubbuck, *On homotopy commutative H-spaces*, Topology **8** (1969), 119-126.

[13] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

[14] I.M. James and E. Thomas, *Homotopy-abelian topological groups*, Topology **1** (1962), 237-240.

[15] D. Kishimoto, *Homotopy nilpotency in localized* SU(n), to appear in Homology, Homotopy and Applications.

[16] A. Kono and H. Ōshima, *Commutativity of the group of self-homotopy classes of Lie groups*, Bull. London Math. Soc. **36** (2004), 37-52.

[17] P.G. Kumpel, *Mod p-equivalences of mod p H-spaces*, Quart. J. Math. **23** (1972), 173-178.

[18] J.P. Lin, *Loops of H-spaces with finitely generated cohomology rings*, Topology Appl. **60** (1994), no. 2, 131-152.

[19] C. McGibbon, *Homotopy commutativity in localized groups*, Amer. J. Math. **106** (1984), 665-687.

[20] C.A. McGibbon, *Higher forms of homotopy commutativity and finite loop spaces*, Math. Z. **201** (1989), no. 3, 363-374.

[21] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Ann. of Math. Studies **76**, Princeton Univ. Press, Princeton N.J., 1974.
[22] M. Mimura and H. Toda, *Topology of Lie Groups I, II*, Translations of Math. Monographs **91**, AMS Providence, RI, 1991.
[23] G. Mislin, *Nilpotent groups with finite commutator subgroups*, Lecture Notes in Math., **418**, Springer, Berlin, 1974, 103-120.
[24] J. Møller, *N-determined 2-compact groups II*, Fund. Math. **196** (2007), no. 1, 1-90.
[25] J. Møller, *N-determined 2-compact groups I*, Fund. Math. **195** (2007), no. 1, 11-84.
[26] L. Saumell, *Homotopy commutativity of finite loop spaces*, Math. Z. **207** (1991), no. 2, 319-334.
[27] J.-P. Serre, *Groupes d'homotopie et classes des groupes abeliens*, Ann. of Math. **58** (1953), 258-294.
[28] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.
[29] C.W. Wilkerson, *K-theory operations and mod p loop spaces*, Math. Z. **132** (1973), 29-44.
[30] A. Zabrodsky, *Hopf Spaces*, North-Holland Mathematics Studies **22**, North-Holland, Amsterdam, 1976.

Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan and Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan