Mod p decompositions of the loops spaces of compact symmetric spaces

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29th British Topology Meeting
Southampton, Sep. 8, 2014

Homotopy Decomposition

When we are given a space X, to classify or to compute its invariants etc, it is convenient to identify X with a product of simpler spaces

$$X \simeq X_1 \times X_2 \times \cdots \times X_n$$

up to homotopy.
Once we have this kind of *homotopy decomposition*, we can calculate, for example, the homotopy groups by

$$\pi_*(X) \simeq \pi_*(X_1) \oplus \cdots \oplus \pi_*(X_n).$$

Mod p decomposition

However, we can hardly expect a homotopy decomposition in general. Then, one possible approach is to consider one prime at a time; instead of attacking X itself, we look at its *localisation* $X_{(p)}$ at a prime p and try to give a decomposition:

$$X_{(p)} \simeq (X_1)_{(p)} \times \cdots \times (X_n)_{(p)}.$$

(We often write $X \simeq_p X_1 \times \cdots X_n$)

We can think of the localisation of a space X as a topological "lift" of the algebraic localisation of the groups $\pi_*(X)$ and $H_*(X)$.
In fact, for simply connected spaces,

$$f : X \simeq_p Y \Leftrightarrow f_* : H_*(X; \mathbb{Z}_{(p)}) \simeq H_*(Y; \mathbb{Z}_{(p)}).$$

Examples

Here's a list of some famous mod p decompositions:

- ([Hopf]) for a finite H-space X, $X \simeq_0 S^{2n_1-1} \times \cdots \times S^{2n_l-1}$.
- ([Serre]) For $p > 2$, $\Omega S^{2n} \simeq_p S^{2n-1} \times \Omega S^{4n-1}$.
- ([Mimura-Nishida-Toda]) For a compact, simple, simply-connected Lie group, there is a prime p (called the *quasi-regular prime for* G) such that
$$G \simeq_{p'} B_1 \times \cdots \times B_n \quad (p' \geq p),$$
where each factor B_i is either an odd sphere S^{2k-1} or a sphere bundle over a sphere $B(2k-1, 2k+2p-3)$;
$$S^{2k-1} \hookrightarrow B(2k-1, 2k+2p-3) \to S^{2k+2p-3}$$

Our goal

Problem

Extend [Mimura-Nishida-Toda] and give mod p decompositions of compact, simply-connected Riemannian symmetric spaces G/K

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However, it is known by [Mimura] that it is impossible in most cases.

So we consider to

Problem
Extend [Serre] and give mod p decompositions of the based loop spaces $\Omega G/K$ of compact, simply-connected Riemannian symmetric spaces

Note that $\pi_*(\Omega X) \simeq \pi_{*+1}(X)$.

Symmetric space

A *Riemannian symmetric space* is a Riemannian manifold M with isometries $\sigma_x : M \to M$ for all $x \in M$ satisfying

$$\sigma_x(x) = x, \quad (d\sigma_x)_x = -1.$$

(Lie groups with $\sigma_x(g) = xg^{-1}x$ are examples)

[Cartan] classified compact, irreducible, simply-connected symmetric spaces (there are 7 classical types and 12 exceptional types).

[Ishitoya-Toda] gave explicit presentations of them as homogeneous spaces G/K, which we rely on for our computation.

Main result

Theorem (K-Ohsita-Theriault)
Let G/K be a compact, simply-connected, Riemannian symmetric space (We take G to be a compact, simply-connected Lie group). Then, $\Omega G/K$ decomposes into a product of spheres, sphere bundles over spheres, and the loops on these spaces after localised at a quasi-regular prime for G.

Recall that an odd prime p is said to be quasi-regular for G if G decomposes into a product of spheres and sphere bundles over spheres after localised at p.
If p is quasi-regular then so is p' $(> p)$.

Sample list

Type	G/K	p (odd)	Homotopy type of $\Omega(G/K)$
AI	$SU(2n+1)/SO(2n+1)$	$p > n$	$\prod_{i=1}^{n-\frac{p-1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\min(1,n-\frac{p-3}{2})}^{\min(n,\frac{p-1}{2})} \Omega S^{4j+1}$
	$SU(4n+2)/SO(4n+2)$	$p = 2n+1$	$\prod_{i=1}^{n-1} \Omega B(4i+1, 4i+2p-1) \times \Omega S^{8n+1} \times \Omega S^{8n+3}$
	$SU(2n)/SO(2n)$	$p > 2n$	$\Omega S^{2n} \times \prod_{i=1}^{n-\frac{p+1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\min(1,n-\frac{p-1}{2})}^{\min(n-1,\frac{p-1}{2})} \Omega S^{4j+1}$
AII	$SU(2n)/Sp(n)$	$p > n$	$\prod_{i=1}^{n-\frac{p+1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\max(1,n-\frac{p-1}{2})}^{\min(n-1,\frac{p-1}{2})} \Omega S^{4j+1}$
$AIII$	$\dfrac{U(n)}{U(m) \times U(n-m)}$	$p > n/2$	$\prod_{j=1}^{m} S^{2j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{2j-1}$
BDI	$\dfrac{SO(2n+1)}{SO(2m) \times SO(2(n-m)+1)}$	$p > n$	$\prod_{j=1}^{m-1} S^{4j-1} \times S^{2m-1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$

The whole list would take more than ten slides...

Applications

(p is always assumed to be a quasi-regular prime for G)

As a direct consequence of the decomposition, we can reduce the calculation of the p-primary part of $\pi_*(G/K)$ to that of $\pi_*(S^{2k-1})$ and $\pi_*(B(2k-1, 2k+2p-3))$.
In particular,

- We can reproduce the computation of the rational homotopy groups of G/K by [Terzic].
- We can give precise values or bounds for p-primary homotopy exponents of G/K: The *p-primary homotopy exponent* of a space X is the least power of p that annihilates the p-torsion in $\pi_*(X)$.

Sample list

Type	G/K	$p(odd)$	Exponent
AI	$SU(2n+1)/SO(2n+1)$	$p > n$	$\begin{cases} \leq p^{4n+2} & \text{if } p-1=n \\ = p^{4n+1} & \text{if } p-1>n \end{cases}$
	$SU(2n)/SO(2n)$	$p > n$	$\begin{cases} \leq p^{4n} & \text{if } p-1=n \\ = p^{4n-1} & \text{if } p-1>n \end{cases}$
AII	$SU(2n)/Sp(n)$	$p > n$	$\begin{cases} \leq p^{4n} & \text{if } p-1=n \\ = p^{4n-1} & \text{if } p-1>n \end{cases}$
$AIII$	$\dfrac{U(n)}{U(m)\times U(n-m)}$	$p > n/2$	$= p^{2n-1}$
BDI	$\dfrac{SO(2n+1)}{SO(2m)\times SO(2(n-m)+1)}$	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+1)}{SO(2m-1)\times SO(2(n-m)+2)}$	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+2)}{SO(2m+1)\times SO(2(n-m)+1)}$	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+2)}{SO(2m)\times SO(2(n-m)+2)}$	$p > n-1$	$= p^{4n-1}$
CI	$Sp(n)/U(n)$	$p > n$	$= p^{4n-1}$
CII	$\dfrac{Sp(n)}{Sp(m)\times Sp(n-m)}$	$p > n$	$= p^{4n-1}$
$DIII$	$SO(2n)/U(n)$	$p > n-1$	$= p^{4n-3}$

Idea of proof

Let p be a quasi-regular prime for G.

1. Homotopy fibration: $\Omega G/K \to K \xrightarrow{\iota} G$
2. $\Omega G/K \simeq \mathrm{fib}(\iota)$, the homotopy fibre of ι
3. By Mimura-Nishida-Toda, $K \simeq_p K_1 \times \cdots \times K_m$ and $G \simeq_p G_1 \times \cdots \times G_n$, where each factor is either a sphere or a sphere bundle over a sphere
4. Decompose ι into a product map $\prod \iota_j : \prod K_j \to \prod G_j$ (this part requires case by case analysis on $\iota^* : H^*(G) \to H^*(K)$)
5. $\mathrm{fib}(\iota) \simeq \prod \mathrm{fib}(\iota_j)$

Homotopy exponent

For the computation of homotopy exponents, the following results are crucial:

Theorem (Cohen-Moore-Neisendorfer)
Let $p \geq 5$. Then $\exp_p(S^{2n+1}) = p^n$

Theorem (Davis-Theriault)
Let $p \geq 5$. Then $\exp_p(B(3, 2p+1)) = p^{p+1}$ and for $k > 2$,

$$p^{k+p-2} \leq \exp_p(B(2k-1, 2k+2p-3)) \leq p^{k+p-1}.$$

Future work

- Extend the decomposition result to smaller primes
- Exploit symmetric space structure to avoid case by case analysis
- Extend the decomposition result to other homogeneous spaces
- Give *stable splittings* of G/K
  (wedge decompositions of the (iterated) suspension of G/K)

Thank you very much !