Samelson products in Sp(2)

Hiroaki Hamanaka

Department of Natural Science, Hyogo University of Teacher Education, Kato, Hyogo 673-1494, Japan

Shizuo Kaji [*,1]

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Akira Kono

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Abstract

We calculate certain Samelson products of Sp(2). Using the result, we classify the homotopy types of the gauge groups of principal Sp(2) bundles over S^8 and we also derive the homotopy commutativity of Sp(2) localized at 3.

Key words: Samelson product, Gauge group, Homotopy commutativity
2000 MSC: 55Q15, 55P10

1 Introduction

For a group-like space G, a generalization of the commutator map γ called the (generalized) Samelson product $\langle f_1, f_2 \rangle$ of two maps $f_1 : Z_1 \to G$ and $f_2 : Z_2 \to G$ is defined as the composition $Z_1 \wedge Z_2 \xrightarrow{f_1 \wedge f_2} G \wedge G \xrightarrow{\gamma} G$.

In this note, we calculate certain Samelson products of Sp(2) and give two applications.

* Corresponding author.
Email addresses: hammer@sci.hyogo-u.ac.jp (Hiroaki Hamanaka), kaji@math.kyoto-u.ac.jp (Shizuo Kaji), kono@math.kyoto-u.ac.jp (Akira Kono).

[1] supported in part by JSPS Postdoctral Fellowships 182641

One is the classification of homotopy types of gauge groups. Let G be a compact Lie group, $\pi : P \to B$ a principal G-bundle over a finite complex B. We denote by $\mathcal{G}_P$, the group of G-equivariant self maps covering the identity map of B. $\mathcal{G}_P$ is called the gauge group of P. If B is connected and G is a compact connected Lie group, in [CS] M.Crabb and W.Sutherland proved that the number of homotopy types of $\mathcal{G}_P$ is finite as P ranges over all principal G-bundles over B. In some situations, exact number of homotopy types are calculated ([K], [HK2], [HK3]).

In this paper we show the following:

Theorem 1 *Denote by ϵ_7' a generator of $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$ and by $\mathcal{G}_k$ the gauge group of principal* $\mathrm{Sp}(2)$ *bundle over S^8 classified by $k\epsilon_7'$ ($k \in \mathbb{Z}$). Then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(140, k) = (140, k')$, where (a, b) denotes the GCD of a and b.*

The other application is on the homotopy commutativity of Sp(2) localized at 3. Although this is already proved by McGibbon ([M]) as a part of his thorough investigation of the homotopy commutativity of localized Lie groups, here we give a new proof which doesn't require any information of the homotopy groups of Sp(2) on which McGibbon relies.

We denote by $-_{(p)}$ the p-localization in the sense of Bousfield and Kan [BK].

Theorem 2 (cf. [M]) $\mathrm{Sp}(2)_{(3)}$ *is homotopy commutative.*

2 Notation

Here we give our notation and some facts which we use throughout this note.

We use the same symbol c' for the inclusion $\mathrm{Sp}(n) \hookrightarrow \mathrm{U}(2n) \hookrightarrow \mathrm{U}(2n+1)$, the complexifications $B\mathrm{Sp}(\infty) \to B\mathrm{U}(\infty)$ and $B\mathrm{Sp}(n) \to B\mathrm{U}(2n+1)$.

Let $W_n = \mathrm{U}(\infty)/\mathrm{U}(n)$, $X_n = \mathrm{Sp}(\infty)/\mathrm{Sp}(n)$, and $\bar{c}' : X_n \to W_{2n+1}$. Then we have the following commutative diagram of fibration sequences

$$\begin{array}{ccccccc}
\mathrm{Sp}(\infty) & \xrightarrow{p'} & X_n & \xrightarrow{i'} & B\mathrm{Sp}(n) & \longrightarrow & B\mathrm{Sp}(\infty) \\
\downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle \bar{c}'} & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c'} \\
\mathrm{U}(\infty) & \xrightarrow{p} & W_{2n+1} & \longrightarrow & B\mathrm{U}(2n+1) & \longrightarrow & B\mathrm{U}(\infty)
\end{array}$$

Let σ be the cohomology suspension. Those facts listed below are well known (see, for example, Chapter 3 of [MT2]):

$$
\begin{aligned}
H^*(B\mathrm{U}(\infty)) &= \mathbb{Z}[c_1, c_2, \ldots]\ (|c_i| = 2i),\\
H^*(B\mathrm{Sp}(\infty)) &= \mathbb{Z}[q_1, q_2, \ldots]\ (|q_i| = 4i),\\
H^*(W_{2n+1}) &= \bigwedge(x'_{4n+3}, x'_{4n+5}, \ldots),\\
H^*(X_n) &= \bigwedge(y'_{4n+3}, y'_{4n+7}, \ldots),\\
c'^*(c_{2j}) &= (-1)^j q_j, c'^*(c_{2j+1}) = 0,\\
p^*(x'_{4n+2j-1}) &= \sigma(c_{2n+j}) = x_{4n+2j-1},\\
p'^*(y'_{4n+4j-1}) &= \sigma(q_{n+j}) = y_{4n+4j-1},\\
\bar{c}'^*(x'_{4n+4j-1}) &= (-1)^{n+j} y'_{4n+4j-1}, \quad \bar{c}'^*(x'_{4n+4j-3}) = 0.
\end{aligned}
$$

Let $a_{4n+2j} = \sigma(x'_{4n+2j+1}), \quad b_{4n+4j-2} = \sigma(y'_{4n+4j-1})$ so that

$$
\begin{aligned}
H^*(\Omega W_{2n+1}) &= \mathbb{Z}\{a_{4n+2}, \ldots, a_{8n+2}\}, \quad (* \leq 8n+2)\\
H^*(\Omega X_n) &= \mathbb{Z}\{b_{4n+2}, \ldots, b_{8n+2}\}, \quad (* \leq 8n+2).
\end{aligned}
$$

We need the following Lemma which gives information on $\Omega p'$.

Lemma 3 *Let Z be a space. For a map $\alpha : \Sigma^2 Z \to B\mathrm{Sp}(\infty)$, we have*

$$
\left(\Omega p' \circ \rho^2 \alpha\right)^* (b_{4n+4j-2}) = (-1)^{n+j}(2n+2j-1)! \cdot \Sigma^2(ch_{2n+2j}(c'(\alpha))),
$$

where ρ is the adjoint and $\Sigma : H^{*+1}(\Sigma Z) \simeq H^*(Z)$ is the suspension isomorphism.

PROOF. Use the equality $(\Omega p)_* a_{4n+4j-2} = (2n+2j-1)! ch_{2n+2j}$ in [HK1].

Here we recall the following Lemma.

Lemma 4 ([N]) *There is a lift $\tilde{\gamma}'$ of the commutator map $\gamma' : \mathrm{Sp}(n) \wedge \mathrm{Sp}(n) \to \mathrm{Sp}(n)$ such that $\delta' \circ \tilde{\gamma}' = \gamma'$ where $\delta' = \Omega i' : \Omega X_n \to \mathrm{Sp}(n)$ and $\tilde{\gamma}'^*(b_{4n+4j-2}) = \sum_{i+m=n+j} y_{4i-1} \otimes y_{4m-1}$.*

Now we specialize to the case when $n = 2$.

Let A be the 7-skeleton of $\mathrm{Sp}(2)$. Then $A = S^3 \cup e^7$ and $\epsilon : A \hookrightarrow \mathrm{Sp}(2)$. Define two maps

$$
\begin{aligned}
a &: \quad \Sigma A \subset \Sigma \mathrm{Sp}(2) \xrightarrow{\rho(1_{\mathrm{Sp}(2)})} B\mathrm{Sp}(2) \to B\mathrm{Sp}(\infty) \text{ and}\\
b &: \Sigma A \to S^8 \xrightarrow{\rho(\epsilon'_7)} B\mathrm{Sp}(2) \to B\mathrm{Sp}(\infty), \text{ where the left most arrow is the projection.}
\end{aligned}
$$

Then we have

$$ch(c'(a)) = \Sigma u_3 - \frac{1}{6}\Sigma u_7 \quad (1)$$
$$ch(c'(b)) = -2\Sigma u_7, \quad (2)$$

where $u_3 = \epsilon^*(y_3)$ (resp. $u_7 = \epsilon^*(y_7)$) is a generator of $H^3(A;\mathbb{Z}) \simeq \mathbb{Z}$ (resp. $H^7(A;\mathbb{Z}) \simeq \mathbb{Z}$).

For a space Z, let $K\mathrm{Sp}(Z)$ be the Hermitian K-theory of Z. Using the short exact sequence

$$0 = \widetilde{K\mathrm{Sp}}^0(S^5) \to \widetilde{K\mathrm{Sp}}^0(S^8) \to \widetilde{K\mathrm{Sp}}^0(\Sigma A) \to \widetilde{K\mathrm{Sp}}^0(S^4) \to \widetilde{K\mathrm{Sp}}^0(S^7) = 0,$$

we have

Lemma 5 *$\widetilde{K\mathrm{Sp}}^0(\Sigma A) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by a and b.*

3 The order of the Samelson product $\left\langle \epsilon'_7, 1_{\mathrm{Sp}(2)} \right\rangle \in [S^7 \wedge \mathrm{Sp}(2), \mathrm{Sp}(2)]$

First we consider the order of the Samelson product $\left\langle \epsilon'_7, \epsilon \right\rangle \in [S^7 \wedge A, \mathrm{Sp}(2)]$.

Lemma 6 *$[\Sigma^7 A, \Omega X_2] \simeq \mathbb{Z} \oplus \mathbb{Z}$.*

PROOF. Recall that $\dim(\Sigma^7 A) = 14, \Omega X_2 = S^{10} \cup e^{14} \cup e^{18} \cdots$. Let F be the homotopy fiber of the inclusion $S^{11} \hookrightarrow X_2$. Then since

$$0 = \pi_{15}(S^{11}) \to \pi_{14}(\Omega X_2) \to \pi_{14}(F) \simeq \mathbb{Z}$$

is exact, we have $\pi_{14}(\Omega X_2) \simeq \mathbb{Z}$. Apply this to the following exact sequence

$$\mathbb{Z}/2 \simeq \pi_{11}(\Omega X_2) \to \pi_{14}(\Omega X_2) \to [\Sigma^7 A, \Omega X_2] \to \pi_{10}(\Omega X_2) \simeq \mathbb{Z},$$

which is associated to the cofibration sequence $\Sigma^7 S^6 \to \Sigma^7 S^3 \to \Sigma^7 A$.

Definition 7 *For $\alpha \in [\Sigma^7 A, \Omega X_2]$, define $\lambda(\alpha) = (\lambda_1(\alpha), \lambda_2(\alpha)) \in \mathbb{Z} \oplus \mathbb{Z}$, where $\alpha^*(b_{10}) = \lambda_1(\alpha)\Sigma^7 u_3$ and $\alpha^*(b_{14}) = \lambda_2(\alpha)\Sigma^7 u_7$.*

Lemma 8 *$\lambda : [\Sigma^7 A, \Omega X_2] \to \mathbb{Z} \oplus \mathbb{Z}$ is a homomorphism and monic.*

PROOF. The map $\xi = y'_{11} \times y'_{15} : X_2 \to K(\mathbb{Z}, 11) \times K(\mathbb{Z}, 15)$ induces a 18 equivalence $\xi_{(0)} : (X_2)_{(0)} \to K(\mathbb{Q}, 11) \times K(\mathbb{Q}, 15)$. Since $\dim \Sigma^7 A = 14$,

$$(\Omega\xi_{(0)})_* : [\Sigma^7 A, (\Omega X_2)_{(0)}] \to H^{10}(\Sigma A; \mathbb{Q}) \oplus H^{14}(\Sigma A; \mathbb{Q})$$

is an isomorphism. By the following commutative diagram

$$\begin{array}{ccccc}
 & & [\Sigma^7 A, \Omega X_2] & \xrightarrow{(\Omega\xi)_*} & H^{10}(\Sigma A) \oplus H^{14}(\Sigma A) \\
 & & \downarrow & & \downarrow \\
[\Sigma^7 A, \Omega X_2]_{(0)} & = & [\Sigma^7 A, (\Omega X_2)_{(0)}] & \xrightarrow[(\Omega\xi_{(0)})_*]{} & H^{10}(\Sigma A; \mathbb{Q}) \oplus H^{14}(\Sigma A; \mathbb{Q}),
\end{array}$$

where vertical arrows are the rationalization maps, we have the lemma since $[\Sigma^7 A, \Omega X_2]$ is free and $\lambda = (\Omega\xi)_*$.

Let $D \in \widetilde{KO}^0(S^8) \simeq \mathbb{Z}$ be the generator (the Dirac element). Then we have $ch(c'(D)) = v_8$, where v_8 is a generator of $H^8(S^8; \mathbb{Z})$.

By Lemma 4, we have the following diagram

$$\begin{array}{ccccc}
 & & & & \Omega\mathrm{Sp}(\infty) \\
 & & & & \downarrow{\scriptstyle \Omega p'} \\
 & & & \overset{\tilde{\gamma}'}{\nearrow} & \Omega X_2 \\
 & & & & \downarrow{\scriptstyle \delta'} \\
S^7 \wedge A & \xrightarrow{\epsilon'_7 \wedge \epsilon} & \mathrm{Sp}(2) \wedge \mathrm{Sp}(2). & \xrightarrow{\gamma'} & \mathrm{Sp}(2)
\end{array}$$

Lemma 9 *$\gamma' \circ (\epsilon'_7 \wedge \epsilon)$ has order* 140.

PROOF. Put $\gamma_1 = \tilde{\gamma}' \circ (\epsilon'_7 \wedge \epsilon)$, $\alpha_1 = (\Omega p')_*(\rho^2(D \hat{\otimes} a))$ and $\beta_1 = (\Omega p')_*(\rho^2(D \hat{\otimes} b))$. Recall that

$$\begin{aligned}
\bar{\gamma}'^*(b_{10}) &= y_3 \otimes y_7 + y_7 \otimes y_3, \\
\bar{\gamma}'^*(b_{14}) &= y_3 \otimes y_{11} + y_7 \otimes y_7 + y_{11} \otimes y_3, \\
(\epsilon'_7)^*(y_7) &= 12 v_7.
\end{aligned}$$

Hence we have $\lambda(\gamma_1) = (12, 12)$.

By Lemma 3, we have $\lambda(\alpha_1) = (5!, \frac{7!}{6}), \lambda(\beta_1) = (0, 2 \cdot 7!)$. Since λ is monic and $\lambda(140\gamma_1 - 14\alpha_1 + \beta_1) = 0$ we get $140\gamma_1 = 14\alpha_1 - \beta_1$.

Consider the following short exact sequence

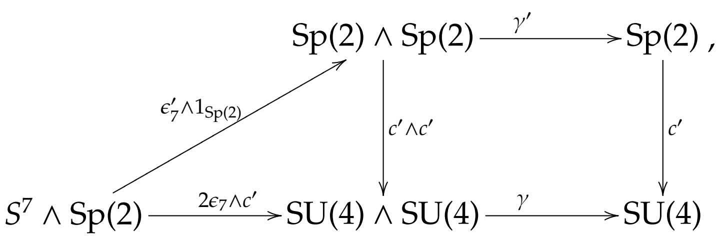
$$0 \to \mathrm{Im}(\Omega p')_* \to [\Sigma^7 A, \Omega X_2] \xrightarrow{\delta'_*} [\Sigma^7 A, \mathrm{Sp}(2)] \to 0. \tag{3}$$

This shows that $140\gamma' \circ (\epsilon'_7 \wedge \epsilon) = 140\delta' \circ \gamma_1 = 14\delta' \circ \alpha_1 - \delta' \circ \beta_1 = 0$.

Proposition 10 *The order of the Samelson product* $\left\langle \epsilon'_7, 1_{Sp(2)} \right\rangle$ *is 140.*

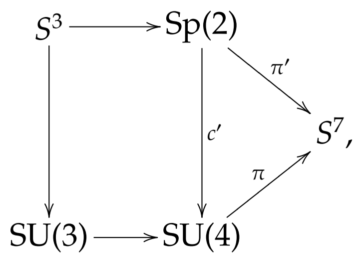
PROOF. Since the attaching map of the top cell of Sp(2) become trivial after double suspension, there exists a map $i : S^{16} \to \Sigma^6\mathrm{Sp}(2)$ such that $S^{17} \vee \Sigma^7 A \xrightarrow{\Sigma i \vee \Sigma^7 \epsilon} \Sigma^7 Sp(2)$ is a homotopy equivalence. Hence we only have to show that $140\gamma_2 = 0$, where $\gamma_2 = \gamma' \circ (\epsilon'_7 \wedge 1_{\mathrm{Sp}(2)}) \circ \Sigma i : S^{17} \to \mathrm{Sp}(2)$.

Let $\epsilon_7 \in \pi_7(\mathrm{SU}(4)) \simeq \mathbb{Z}$ be a generator. Since $c'_*(\epsilon'_7) = 2\epsilon_7$, we have the following commutative diagram



where $\gamma : \mathrm{SU}(4) \wedge \mathrm{SU}(4) \to \mathrm{SU}(4)$ is the commutator.

By the map of fibrations



we have $\pi' \circ \gamma_2 = \pi \circ c' \circ \gamma_2 = 2\pi \circ \gamma \circ (\epsilon_7 \wedge c')$. Since

$$\pi'_* : \mathbb{Z}/8 \oplus \mathbb{Z}/5 \simeq \pi_{17}(\mathrm{Sp}(2)) \to \pi_{17}(S^7) \simeq \mathbb{Z}/24 \oplus \mathbb{Z}/2$$

induces an injection on 2-primary part by Mimura-Toda [MT1], we have $20\gamma_2 = 0$.

PROOF. [Proof of Theorem 1] By [AB], the classifying space $B\mathcal{G}_P$ of the gauge group $\mathcal{G}_P$ of a principal G-bundle P over a finite complex B, is homotopy equivalent to $\mathrm{Map}_P(B, BG)$, the connected component of maps from B to BG containing the classifying map of P. Consider the fibre sequence arose from the evaluation fibration

$$\mathcal{G}_k \to \mathrm{Sp}(2) \xrightarrow{h_k} \mathrm{Map}^*_{k\epsilon'_7}(S^8, B\mathrm{Sp}(2)) \to \mathrm{Map}_{k\epsilon'_7}(S^8, B\mathrm{Sp}(2)) \xrightarrow{e_k} B\mathrm{Sp}(2). \quad (4)$$

By Lang [L] $\mathrm{Map}^*_{k\epsilon'_7}(S^8, B\mathrm{Sp}(2))$ is homotopy equivalent to $\mathrm{Map}^*_0(S^8, B\mathrm{Sp}(2))$ and h_k can be identified with $\left\langle 1_{\mathrm{Sp}(2)}, k\rho\epsilon'_7 \right\rangle = k\left\langle 1_{\mathrm{Sp}(2)}, \rho\epsilon'_7 \right\rangle = kh_1$ in

$$[\mathrm{Sp}(2), \mathrm{Map}^*_0(S^8, B\mathrm{Sp}(2))] \cong [\Sigma^8\mathrm{Sp}(2), B\mathrm{Sp}(2)] \cong [\Sigma^7\mathrm{Sp}(2), \mathrm{Sp}(2)].$$

Here we follow the argument in [HK2] briefly, which completes the proof.

Since $\pi_*(\Omega^7_0\mathrm{Sp}(2))$ is finite for each degree, we have $\Omega^7_0\mathrm{Sp}(2) = \prod \Omega^7_0\mathrm{Sp}(2)_{(p)}$. Let $\nu_p(n)$ be the exponent of n at a prime p. When $(140, k) = (140, k')$, we can define a self homotopy equivalence $\left(\frac{k'}{k}\right)_{140}$ of $\Omega^7_0\mathrm{Sp}(2)$ by

$$\begin{cases} \left(\frac{k'}{k}\right) & \text{if } \nu_p(k) < \nu_p(140) \\ 1 & \text{if } \nu_p(k) \geq \nu_p(140), \end{cases}$$

on each factor $\Omega^7_0\mathrm{Sp}(2)_{(p)}$. Since $140h_1 = 0$, we have $\left(\frac{k'}{k}\right)_{140} \circ (kh_1) \simeq k'h_1$. Therefore if $(140, k) = (140, k')$, then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$. Conversely, applying the functor $[A, \ ]$ to (4) and comparing it to (3), we have $(140, k) = (140, k')$ if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$.

4 The order of the Samelson product $\left\langle 1_{\mathrm{Sp}(2)}, 1_{\mathrm{Sp}(2)} \right\rangle$

In [M] McGibbon showed that $\mathrm{Sp}(2)_{(3)}$ is homotopy commutative. Here we give another proof of this fact.

Denote the mod 3 reduction of y'_{4j+3} $(j \geq 2)$ by the same symbol. Then we have $H^*(X_2; \mathbb{Z}/3) = \bigwedge(y'_{11}, y'_{15}, y'_{19}, \ldots)$ and $\mathcal{P}^1 y'_{11} = \pm y'_{15}$, where $\mathcal{P}^1$ is the first mod 3 Steenrod operation. Let E be the homotopy fiber of $(\beta\mathcal{P}^1 u_{11})_{(3)} : K(\mathbb{Z}_{(3)}, 11) \to K(\mathbb{Z}_{(3)}, 16)$, where u_{11} is a generator of $H^{11}(K(\mathbb{Z}, 11); \mathbb{Z}/3)$. Since $\mathcal{P}^1 y'_{11} = \pm y'_{15}$ and $\beta\mathcal{P}^1 y'_{11} = 0$, the map $(y'_{11})_{(3)} : (\Omega X_2)_{(3)} \to K(\mathbb{Z}_{(3)}, 11)$ lifts to a 17 equivalence $f : (\Omega X_2)_{(3)} \to E$. Since $\dim(A \wedge A) = 14$,

$$(\Omega f)_* : [A \wedge A, (\Omega X_2)_{(3)}] = [A \wedge A, \Omega X_2]_{(3)} \to [A \wedge A, \Omega E]$$

is an isomorphism of groups. Consider the following exact sequence:

$H^9(A\wedge A;\mathbb{Z}_{(3)}) \to H^{14}(A\wedge A;\mathbb{Z}_{(3)}) \to [A\wedge A,\Omega E] \to H^{10}(A\wedge A,\mathbb{Z}_{(3)}) \to H^{15}(A\wedge A;\mathbb{Z}_{(3)})$.

Since $H^k(A\wedge A;\mathbb{Z}_{(3)}) = \begin{cases} 0 & k=9,15 \\ \mathbb{Z}_{(3)} & k=10,14, \end{cases}$ we have $[A\wedge A,\Omega E]\simeq \mathbb{Z}_{(3)}\oplus\mathbb{Z}_{(3)}$.

Define $\tilde{\lambda} : [A\wedge A,\Omega X_2]_{(3)} \to (\mathbb{Z}_{(3)})^3$ by $\tilde{\lambda}(\alpha) = (\tilde{\lambda}_1(\alpha),\tilde{\lambda}_1'(\alpha),\tilde{\lambda}_2(\alpha))$ where $\alpha^*(\Omega c')^*(a_{10}) = \tilde{\lambda}_1(\alpha)u_3\otimes u_7 + \tilde{\lambda}_1'(\alpha)u_7\otimes u_3$ and $\alpha^*(\Omega c')^*(a_{14}) = \tilde{\lambda}_2(\alpha)u_7\otimes u_7$ for $\alpha\in[A\wedge A,\Omega X_2]_{(3)}$. Since $\tilde{\lambda}_{(0)} : [A\wedge A,\Omega X_2]_{(0)} \to (\mathbb{Q})^3$ is an isomorphism (see section 3), $\tilde{\lambda}$ is monic.

It is not hard to show $c' : \widetilde{KSp}(\Sigma^2 A\wedge A)_{(3)} \to \tilde{K}(\Sigma^2 A\wedge A)_{(3)}$ is an isomorphism. Therefore we may consider $a\hat{\otimes}a, a\hat{\otimes}b + b\hat{\otimes}a \in \widetilde{KSp}(\Sigma^2 A\wedge A)_{(3)}$.

Put $\alpha_1 = \dfrac{6}{5!}(\Omega p')_*(\sigma^2(a\hat{\otimes}a))$ and $\alpha_2 = \dfrac{9}{2\cdot 6!}(\Omega p')_*(\sigma^2(a\hat{\otimes}b + b\hat{\otimes}a))$. Then $\alpha_1,\alpha_2\in [A\wedge A,\Omega X_2]_{(3)}$. Using equalities $ch(c'(a)) = \Sigma u_3 - \dfrac{1}{6}\Sigma u_7$ and $ch(c'(b)) = -2\Sigma u_7$, we can easily show

$$\tilde{\lambda}(\alpha_1) = (1,1,-7), \tilde{\lambda}(\alpha_2) = (3,3,-42).$$

By the same method as in the proof of Lemma 9, we have

$$\tilde{\lambda}(\tilde{\gamma}'\circ(\epsilon\wedge\epsilon)) = (-1,-1,1).$$

Since $\tilde{\lambda}(\tilde{\gamma}'\circ(\epsilon\wedge\epsilon) + \alpha_1 + \dfrac{2}{7}(3\alpha_1-\alpha_2)) = 0$ and $\tilde{\lambda}$ is monic, we have

Lemma 11 $\gamma'\circ(\epsilon\wedge\epsilon) = 0$ *in* $[A\wedge A,\mathrm{Sp}(2)]_{(3)}$.

PROOF. [Proof of Theorem 2] Since $\Sigma A \hookrightarrow \Sigma\mathrm{Sp}(2)$ has a retraction, we have to show that the generalized Whitehead product $\Sigma A\wedge\Sigma A \to B\mathrm{Sp}(2)$ vanish. Taking the adjoint, previous Lemma completes the proof.

References

[AB] M.F.Atiyah and R.Bott, *The Yang-Mills equations over Riemann surfaces*, Phils. Trans. Ray. Soc. London Ser A., **308** (1982), 523-615.

[BK] A.K. Bousfield, D.M. Kan, *Homotopy Limits, Completions and Localizations*, LNM **304** (1972), Springer-Verlag, Berlin-New York.

[CS] M.C.Crabb and W.A.Sutherland, *Counting homotopy types of gauge groups*, Proc. London. Math. Soc., (3) **81** (2000), 747-768.

[H] H.Hamanaka, *On* $[X, \mathrm{U}(n)]$ *when* $\dim X = 2n+1$, J. Math. Kyoto Univ. **44** (2004), no. 3, 655–667.

[HK1] H.Hamanaka and A.Kono, *On* $[X, \mathrm{U}(n)]$ *when* $\dim X = 2n$, J. Math. Kyoto Univ., **42** (2003) no. 2, 333–348.

[HK2] H.Hamanaka and A.Kono, *Unstable* K^1*-group and homotopy type of certain gauge groups*. Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), no. 1, 149–155.

[HK3] H.Hamanaka and A.Kono, *Homotopy type of gauge groups of* SU(3)*-bundles over* S^6. Topology Appl. **154** (2007), no. 7, 1377–1380.

[K] A.Kono, *A note on the homotopy type of certain gauge groups*, Proc. Royal Soc. Edinburgh, **117A** (1991), 295-297.

[L] G.E.Lang, *The evaluation map and EHP sequence*, Pacific J. Math., 44(1973), 201-210.

[M] C.McGibbon, *Homotopy commutativity in localized groups*, Amer. J. Math. **106** (1984), 665-687.

[MT1] M.Mimura and H.Toda, *Homotopy groups of symplectic groups*, J. Math. Kyoto Univ. **3** (1964), 251–273.

[MT2] M. Mimura and H. Toda, *Topology of Lie groups. I, II*, Transl. Math. Monogr. **91**, American Mathematical Society, Providence, RI, 1991.

[N] T.Nagao, *On the groups* $[X, \mathrm{Sp}(n)]$ *with* $\dim X \leq 4n+2$, to appear in J. Math. Kyoto Univ.